THE ERDŐS-GYÁRFÁS FUNCTION $f(n,4,5)=\frac{5}{6}n+o(n)$ — SO GYÁRFÁS WAS RIGHT

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ABSTRACT. A $(4,5)$-coloring of K_n is an edge-coloring of K_n where every 4-clique spans at least five colors. We show that there exist $(4,5)$-colorings of K_n using $\frac{5}{6}n+o(n)$ colors. This settles a disagreement between Erdős and Gyárfás reported in their 1997 paper. Our construction uses a randomized process which we analyze using the so-called differential equation method to establish dynamic concentration. In particular, our coloring process uses random triangle removal, a process first introduced by Bollobás and Erdős, and analyzed by Bohman, Frieze and Lubetzky.

1. INTRODUCTION

In 1975, Erdős and Shelah [13] defined the following generalization of classical Ramsey numbers.

Definition 1. *Fix integers p,q such that $p\ge 3$ and $2\le q\le \binom{p}{2}$. A (p,q)-*coloring *of K_n is a coloring of the edges of K_n such that every p-clique has at least q distinct colors among its edges. The Erdős-Gyárfás function $f(n,p,q)$ is the minimum number of colors such that K_n has a (p,q)-coloring.*

We are interested in fixing p,q and investigating the asymptotic behavior of $f(n,p,q)$ as n tends to infinity. In particular we will be investigating $f(n,4,5)$. But in order to introduce the general problem, we will discuss what is known about other "small" pairs (p,q). We start with the case where $q=2$, which is equivalent to a classical Ramsey problem. Recall that we define the Ramsey number $R_k(p)$ to be the smallest natural number N such that every edge-coloring of K_N using k colors yields a monochromatic p-clique. Thus, $f(n,p,2)$ is the smallest k such that $R_k(p)>n$. The following lower bound was proved by Lefmann [23] and the upper bound follows from the Erdős-Szekeres "neighborhood chasing" argument [16]:

$$2^{kp/4}\le R_k(p)\le k^{kp}.$$

It follows for fixed $p\ge 3$ that

$$\Omega\left(\frac{\log n}{\log\log n}\right)=f(n,p,2)=O(\log n).$$

Next we discuss $(3,3)$-colorings. This case is easy but we would like to use it to preview $(4,5)$-colorings. It is not difficult to see that $f(n,3,3)=\chi'(K_n)$, since a $(3,3)$-coloring is precisely a proper edge coloring of K_n, i.e. a decomposition of the edges into matchings. Later we will see that finding $f(n,4,5)$ also involves a type of decomposition problem (with additional

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constraints). Using the well-known values of $\chi'(K_n)$ we get

$$f(n,3,3)=\chi'(K_n)=\begin{cases} n-1, & n \text{ is even,}\\ n, & n \text{ is odd.}\end{cases}$$

We now consider (p,q)-colorings where $p \geq 4$ and $q \geq 3$. It is easy to see that here we have $f(n,p,\binom{p}{2})=\binom{n}{2}$. Thus if we examine the sequence of functions $f(n,p,2), f(n,p,3),\ldots,$ $f(n,p,\binom{p}{2})$ we see it starts with at most logarithmic growth and gets larger until we see quadratic growth. Erdős and Gyárfás [15] found for each p the smallest value of q such that $f(n,p,q)$ is at least linear in n (such q is called the *linear threshold*). They also found the smallest q such that $f(n,p,q)$ is quadratic (the *quadratic threshold*). In particular, they showed that the linear threshold is $q=\binom{p}{2}-p+3$ and that the quadratic threshold is $q=\binom{p}{2}-\lfloor p/2\rfloor+2$. Among several other questions posed in [15], they ask the following: for fixed p, what is the smallest q such that $f(n,p,q)$ is polynomial in n (the *polynomial threshold*)? They showed that the polynomial threshold for any p is at most p, and in particular

$$f(n,p,p)\geq n^{\frac{1}{p-2}}. \tag{1}$$

For $(4,3)$-colorings, the following lower bound is due to Fox and Sudakov [18] and upper bound is due to Mubayi [24]:

$$\Omega(\log n)=f(n,4,3)\leq \exp\left\{O\left(\sqrt{\log n}\right)\right\}=n^{o(1)}.$$

Thus, the polynomial threshold for $p=4$ is $q=4$.

For $(4,4)$-colorings, the following lower bound follows from equation (1) and upper bound is due to Mubayi [25]:

$$n^{1/2}\leq f(n,4,4)\leq n^{1/2}\exp\left\{O\left(\sqrt{\log n}\right)\right\}=n^{1/2+o(1)}.$$

Thus, we arrive at $(4,5)$-colorings, which is the main focus of this paper. Of course $f(n,4,5)=\Omega(n)$ since $q=5$ is the linear threshold for $p=4$. Moreover, Erdős and Gyárfás [15] paid special attention to $f(n,4,5)$ and gave a proof that

$$\frac{5}{6}(n-1)\leq f(n,4,5)\leq n,$$

although the lower bound was previously stated by Erdős, Elekes and Füredi [14]. Since the coefficients $5/6$ and 1 are so close, Erdős and Gyárfás were tempted to make a guess as to what the true coefficient should be. Erdős thought that it should be 1, while Gyárfás thought that it should be "closer to 5/6" [15]. Our main theorem settles this disagreement:

Theorem 1. *We have*

$$f(n,4,5)=\frac{5}{6}n+o(n).$$

Let us also mention that the function $f(n,p,q)$ has been extensively studied by several other researchers, see, e.g., [2, 3, 9, 10, 11, 12, 17, 26, 27, 28].

1.1. **Proof overview.** We outline the proof of Theorem 1. The lower bound was proved by Erdős and Gyárfás [15], and for the sake of completeness we will restate their proof in Section 2. For the upper bound, it clearly suffices to show that for any fixed $\varepsilon>0$ we have $f(n,4,5)\leq \frac{5}{6}n+\varepsilon n$ for all sufficiently large n. We show that there exists some randomized coloring procedure using $\frac{5}{6}n+\varepsilon n$ colors such that the probability of getting a $(4,5)$-coloring is positive for sufficiently large n. We will then define a procedure using two *phases*. The *first phase* will (if successful) use $\frac{5}{6}n+\frac{1}{2}\varepsilon n$ colors to color almost all the edges of K_n using a randomized coloring process, and the analysis of this phase will be the main work of this paper. The *second phase* will color the remaining uncolored edges using a much simpler random coloring and a fresh set of $\frac{1}{2}\varepsilon n$ colors. Our analysis of the first phase of the process will show that with positive probability it outputs a partial coloring with nice properties that will allow us to easily show that the second phase successfully finishes a $(4,5)$-coloring with positive probability, which completes the proof.

For the first phase we will use the differential equation method (see [4] for a gentle introduction) to establish dynamic concentration of our random variables. The origin of the differential equation method stems from work done at least as early as 1970 (see Kurtz [22]), and which was developed into a very general tool by Wormald [30, 31] in the 1990's. Indeed, Wormald proved a "black box" theorem, which gives dynamic concentration so long as some relatively simple conditions hold. Warnke [29] recently gave a short proof of a somewhat stronger black box theorem. For our purposes the existing black box theorems are insufficient, but we are still able to analyze our process using fairly standard arguments that resemble previous analyses of other processes.

The analysis of the second phase will be based on the Lovász Local Lemma.

1.2. **Tools.** We will be using the following forms of Chernoff's bound (see, e.g., [21]).

Lemma 1 (Chernoff bound). *Let $X\sim Bin(n,p)$ and $\mu=E(X)=np$. Then, for all* $0<\delta<1$

$$\Pr(X\ge(1+\delta)\mu)\le\exp(-\mu\delta^2/3) \tag{2}$$

and

$$\Pr(X\le(1-\delta)\mu)\le\exp(-\mu\delta^2/2). \tag{3}$$

We will also need Freedman's inequality [19], which we state next.

Lemma 2 (Freedman's inequality). *Let $W(i)$ be a supermartingale with $\Delta W(i)\le D$ for all i, and let*
$V(i):=\sum_{k\le i}\mathbf{Var}[\Delta W(k)|\mathcal{F}_k]$. *Then,*

$$\mathbb{P}[\exists i: V(i)\le b, W(i)-W(0)\ge\lambda]\le\exp\left(-\frac{\lambda^2}{2(b+D\lambda)}\right).$$

Finally, let us state the Lovász Local Lemma (LLL) [1]. For a set of events $\mathscr{A}$ and a graph G on vertex set $\mathscr{A}$, we say that G is a *dependency graph* for $\mathscr{A}$ if each event $A\in\mathscr{A}$ is not adjacent to events which are mutually independent.

Lemma 3 (Lovász Local Lemma). *Let $\mathscr{A}$ be a finite set of events in a probability space Ω and let G be a dependency graph for $\mathscr{A}$. Suppose there is an assignment $x:\mathscr{A}\to[0,1)$ of real numbers to $\mathscr{A}$ such that for all $A\in\mathscr{A}$ we have*

$$\Pr(A)\le x(A)\prod_{B\in N(A)}(1-x(B)). \tag{4}$$

Then, the probability that none of the events in $\mathscr{A}$ happen is

$$\Pr\left(\bigcap_{A\in\mathscr{A}}\overline{A}\right)\ge\prod_{A\in\mathscr{A}}(1-x(A))>0.$$

1.3. **Organization of the paper.** In Section 2 we motivate and formally define the process for the first phase of our coloring procedure. Our analysis of the process will hinge on our ability to maintain good estimates of a family of random variables which change with each step of the process. In Section 3 we define our family of variables, and in Section 4 we state the bounds that we intend to prove for our random variables. In Sections 6–11 we bound the probability that any of our random variables violate the stated bounds until almost all the edges are colored. This will finish the first phase of the proof, which leaves just a few uncolored edges. Finally, in Section 12 we show how to color such uncolored edges.

2. The coloring process

First we give some motivation. Let us start by seeing the proof of the lower bound for Theorem 1, which was given by Erdős and Gyárfás [15]. The proof will illuminate what needs to be done to achieve an asymptotically matching upper bound (and we will comment on that after the proof).

Theorem 2 ([15]). *We have*

$$f(n,4,5) \geq \frac{5}{6}(n-1).$$

Proof. Suppose we have a $(4,5)$-coloring of a graph of order n using a set of colors C. For each $c \in C$ let G_c be the graph of order n with only the c-colored edges. Note that G_c can never have a connected component with more than two edges (i.e. all components have at most three vertices and there are no monochromatic triangles). Thus every component of G_c is either P^0, P^1, or P^2, where P^j denotes a path on j edges. For $0 \leq j \leq 2$ let x_j be the total number of components P^j in all the graphs G_c, $c \in C$. Thus

$$x_0 + 2x_1 + 3x_2 = n|C| \tag{5}$$

and

$$x_1 + 2x_2 = \binom{n}{2}. \tag{6}$$

Note also that whenever we have a component in color c with two edges on three vertices (i.e. a component counted by x_2), the third edge in that triangle must be a component counted by x_1. Thus, $x_1 \geq x_2$, and hence, $x_0 + \frac{1}{3}(x_1 - x_2) \geq 0$. But then, using (5) and (6), we get

$$|C| = \frac{x_0 + 2x_1 + 3x_2}{n} \geq \frac{x_0 + 2x_1 + 3x_2 - \big(x_0 + \frac{1}{3}(x_1 - x_2)\big)}{n} = \frac{\frac{5}{3}(x_1 + 2x_2)}{n} = \frac{5}{6}(n-1).$$

□

From the proof we can see that the only way to achieve equality would be if $x_0 = 0$ and $x_1 = x_2$. Erdős [15] expressed doubt that any coloring could come close to that. Indeed, he suspected that if we have $x_0 = o(n^2)$ then we must also have $x_2 = o(n^2)$, i.e. x_1 dominates everything and essentially all the graphs G_c are matchings with $n/2 - o(n)$ edges. Such a coloring would have $|C| = n - o(n)$. Indeed, Erdős and Gyárfás [15] gave such a coloring to prove the upper bound $f(n,4,5) \leq n$.

To prove Theorem 1, we will need to get a coloring with $x_0 = o(n^2)$ and $x_1 = x_2 + o(n^2)$. In other words, for almost every P^1 component in some graph G_c, its two endpoints are also the ends of some P^2 component in some $G_{c'}$. Thus we are motivated to consider a process which at each step i colors the edges of some triangle T_i (whose edges have no colors yet), giving two of them the same color and the third one a different color. The intent is to create one P^1

component in a color c and a P^2 component in another color c'. When a vertex v is incident to an edge of some color c we say v has been *hit* by c. To ensure that our components do not accidentally become larger than intended, at each step we will have to make sure to choose c, c' that have not already hit the vertices they are about to hit.

There are many ways we could choose the triangle T_i whose edges we will color at step i. We will use what seems to be the most natural (and well-studied) candidate: the *random triangle removal process* first introduced by Bollobás and Erdős (see [7, 8]). In this process one starts with $G_R(0)=K_n$ and at each step i removes the edges of one triangle chosen uniformly at random from all triangles in $G_R(i)$, stopping only when the graph becomes triangle-free. Bollobás and Erdős conjectured that the number of edges remaining at the end of this process (i.e. edges not in the triangle packing) is $\Theta(n^{3/2})$ a.a.s. (*asymptotically almost surely*, that is, with probability tending to one as $n\to\infty$). The best known estimate (both upper and lower bounds) on the number of edges remaining is $n^{3/2+o(1)}$ by Bohman, Frieze and Lubetzky [6]. We will not need the full power of their result, but for our convenience we will use a few facts they proved in their analysis of the process.

For our coloring process, at each step i we will choose our triangle T_i uniformly at random from all triangles whose edges are all uncolored. We will then randomly choose an *orientation* for T_i, meaning that we choose which of the three edges will be in a P^1 component (meaning the other two will make a P^2). We will then randomly choose two "suitable" colors c_i, c_i' to assign to the edges of T_i. In the end we will use a somewhat complicated rule to determine which colors are "suitable" here. Of course our rule must not violate the constraint for $(4,5)$-coloring, which requires for each set of four vertices to have five different colors among its six edges. So somehow our process must prevent the creation of any set of four vertices having two repeated colors (or one color repeated twice).

Suppose $T_i=\{u,u',u''\}$ is the selected triangle. We will choose the orientation such that $u'u''$ will be assigned the color c_i' and the other two edges will be assigned c_i. In this case, we say that the triangle is oriented *away from u*. Obviously, by our previous discussion we should choose the colors such that c_i' has not hit u' or u'' and c_i has not hit u, u', or u''. Thus throughout the process our coloring will have no color components with more than two edges and our color components P^1 and P^2 will come in pairs sharing endpoints. This requirement already avoids many of the ways our coloring could violate the constraint for a $(4,5)$-coloring. For example, since our color components have at most two edges, we cannot have four vertices containing three edges of the same color. Thus any violation of a $(4,5)$-coloring must involve two different colors, each appearing twice in the same set of four vertices. The rule we have already described also avoids the two configurations illustrated in Figure 1.

FIGURE 1. Two possible configurations that would violate a $(4,5)$-coloring.

However, unless we impose some additional rules for choosing our colors, our process would allow the configurations depicted in Figure 2 that would violate the constraint for a $(4,5)$-coloring.

FIGURE 2. Two additional configurations that would violate a $(4,5)$-coloring and require extra consideration.

We will have to impose two more rules to avoid these violations. Of course, we are still trying to use only $\frac{5}{6}n + o(n)$ colors. We pause to comment that based on the rules we have stated so far, it is heuristically plausible that we could use the process to color almost all edges using $\frac{5}{6}n + o(n)$ colors. Indeed, after $i \le \frac{1}{3}\binom{n}{2}$ steps there are i colored triangles, and so each vertex v should be incident with about $3i/n$ of them. About $1/3$ of those triangles will get oriented in a way that means v gets hit by only one color whereas $2/3$ of these triangles have v getting hit by two colors. Thus the number of colors that have hit v should be about

$$\frac{3i}{n} \cdot \left[\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2\right] = \frac{5i}{n} \le \frac{5 \cdot \frac{1}{3}\binom{n}{2}}{n} \le \frac{5}{6}n$$

and so, heuristically, given any vertex it should be possible to choose a color that has not hit it yet (until almost all edges are colored). However, running this simpler process might require substantial extra colors to make it into a $(4,5)$-coloring afterwards. Thus, we impose the following additional rules into our process.

We will avoid the configuration in Figure 2a (an *alternating 4-cycle*) by "brute force" adjustment: when we choose our colors we will simply refuse to create such a cycle (i.e. color choices that would create one are eliminated from consideration and we randomly choose from the remaining colors). While this rule does make the process more challenging to analyze, we will see that it does not reduce the number of choices we have for colors too significantly.

To avoid the configuration in Figure 2b, it is tempting to say that we will use "brute force" again and simply refuse to make it. However, some thought reveals that this idea is not too promising if we want to use only $\frac{5}{6}n + o(n)$ colors. Indeed, when we have colored, say, about half of the edges, a vertex v should be in some linear number of triangles oriented away from v. Unless our process has a rule to prevent it, we would expect to see some linear number of colors (like the colors $c, \dots, c'$ in Figure 3) appearing across from v in those triangles. None of those colors can be allowed to hit v, since then we would get Figure 2b. However, the simpler process (without additional rules) was using very close to $\frac{5}{6}n$ colors and to come close to $\frac{5}{6}n$ colors we need to make sure almost every vertex gets hit by almost every color. Thus, this proposed "brute force" rule is not viable.

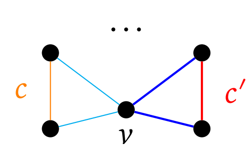


FIGURE 3. If using a "brute force" adjustment to the process, there would be a linear number of colors $c, \dots, c'$ across from v.

In order to overcome this issue, we will do the following. Each vertex v will have some small linear set of *special colors* S_v, which will be the only colors we allow to appear opposite from v in triangles oriented away from v. To avoid the configuration in Figure 2b, we will make sure that v is never hit by any color in S_v.

2.1. **First phase.** In this subsection we define the random process we will use to color almost all edges.

Suppose we have a set $\overline{\text{COL}}$ of $\frac{5}{6}n + \varepsilon n$ colors for some $0 < \varepsilon < 1/100$. Our coloring method has two phases and the first phase will need almost all the colors. We let $\text{COL} \subseteq \overline{\text{COL}}$ be some subset of $\frac{5}{6}n + \frac{\varepsilon}{2}n$ colors. We will use colors in COL for the first phase and reserve the rest for the second phase. We will now start describing the first phase in detail, which we motivated before this subsection.

First, independently for each vertex v and color k, we put k into the set S_v with probability

$$s := \frac{\frac{\varepsilon}{2}}{\frac{5}{6} + \frac{\varepsilon}{2}}.$$

The colors in S_v will not be allowed to hit v, and they will be the only colors allowed to appear across from v in triangles oriented away from v. Note that s was chosen so that the number of colors we allow to hit v, i.e. $|\text{COL} \setminus S_v|$, has expectation $\frac{5}{6}n$.

We will need the following definitions.

Definition 2. *An* alternating (uv,k)-path *is a $u-x-y-v$ path such that edges ux and vy are colored the same color and edge xy is colored k.*

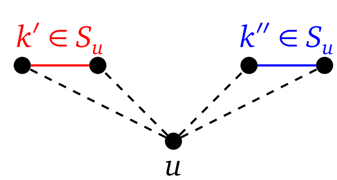
Definition 3. *Let k and k' be colors in* COL.

- *We say k is* available at a vertex *u at step i if $k \notin S_u$, and u has not been hit by k.*
- *We say k is* available at an edge *uv at step i if uv is uncolored, k is available at each of the vertices u and v, and there is no alternating (uv,k)-path.*
- *We say k' is* 1-available at a triple *(u,u',u'') at step i if $k' \in S_u$ and k' is available at the edge $u'u''$.*
- *We say k is* 2-available at a triple *(u,u',u'') at step i if k is available at the edges uu' and uu''. We say a pair (k,k') is* available at a triple *(u,u',u'') at step i if k' is 1-available at (u,u',u'') and k is 2-available at (u,u',u'').*

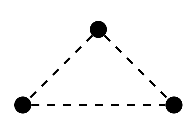
(Note that this definition implies all edges in $uu'u''$ are uncolored. Also, the roles of u' and u'' are interchangeable but the role of u is different).

Now, we are ready to define the process.

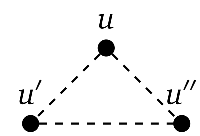
Substep 1. (Initialization) Start with a complete, uncolored graph $K_n = (V,E)$ on n vertices. Recall that for each $u \in V$, we have a set S_u of colors from COL. Colors from S_u can be assigned as the opposite color to u when the triangle is oriented away from u (see opposite figure). On the other hand, these colors are not not allowed to touch u.



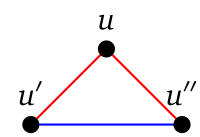
Substep 2. (Triangle) At step i, choose an uncolored triangle T_i uniformly at random from the set of uncolored triangles.



Substep 3. (Orientation) Choose one vertex u uniformly at random from the three vertices of the selected triangle T_i. The triangle will be oriented away from u. Label the other vertices u' and u''.



Substep 4. (Color the triangle) Choose a pair of colors (k, k') uniformly at random from all pairs that are available at triple (u, u', u'') (or terminate if there is no such pair). Note that we can choose k and k' independently from each other. More specifically, we choose k' uniformly at random from all colors such that $k' \in S_u$ and k' is available at $u'u''$. Independent from the choice of k' we choose $k \notin S_u$ uniformly at random from all colors such that k is available at both uu' and uu''. Color uu' and uu'' with k and $u'u''$ with k'.



Substep 5. If there are more uncolored triangles, then go back to Substep 2 and carry out step $i + 1$. Otherwise, terminate.

Note that there are two possible endings of the process: it could finish at Substep 4 because no pair of colors is available, or it could finish at Substep 5 because no uncolored triangles remain. Bohman, Frieze and Lubetzky [6] showed that the random triangle removal process a.a.s. does not terminate until $n^{3/2+o(1)}$ edges remain, and so our process a.a.s. does not terminate at Substep 5 until the number of uncolored edges is $n^{3/2+o(1)}$. Most of this paper is devoted to showing that a.a.s. our coloring process does not terminate at Substep 4 until we have colored almost all the edges. When the process terminates, we will we move to the second phase of our coloring procedure, which will assign colors to the remaining uncolored edges.

We note that some similar ideas were used by Guo, Patton and Warnke [20]. In particular they used a coloring process assigning colors one at a time where each color was chosen uniformly at random from all "available" colors (for some appropriate definition of "available").

2.2. **Second phase.** The second phase will use the set of $\frac{\varepsilon}{2}n$ *reserved colors* $\overline{\text{COL}} \setminus \text{COL}$. Each edge that still needs to be colored will get one of the reserved colors chosen uniformly at random. Our analysis of the first phase will show that it produces a partial coloring that enjoys several useful "pseudorandom" properties (i.e. properties that one would expect to see in a simpler random coloring where each edge has an independent random color). These properties will allow us to argue that the remaining edges are relatively easy to color. We will use the Lovász Local Lemma to show that with positive probability the resulting coloring is a $(4, 5)$-coloring and so, by the trivial probabilistic method, there exists an appropriate extension of the partial coloring we produced in the first phase to a complete $(4, 5)$-coloring.

3. System of random variables

Our analysis of the process in the first phase will proceed by the differential equation method. As usual, we will define a family of random variables which we will *track* throughout the process, meaning that we will obtain asymptotically tight estimates which hold a.a.s.. For each tracked variable there will be a deterministic function, called the *trajectory*, such that a.a.s. the tracked variable is asymptotically equal to its trajectory. Our family of variables will also include some for which we prove only crude upper bounds (but which we do not track).

A beautiful aspect of the differential equation method is that often the trajectories of random variables can be guessed using the right intuition and heuristics. Fortunately we will see that this is the case for our process. Indeed, our family of random variables will have elementary trajectories which we can guess using heuristics. In the next subsection we describe these heuristics, and in the following subsection we define our random variables. As we define each variable we state its trajectory.

3.1. **Heuristics.** Before we start listing the random variables, let us go over the heuristic assumptions. We define the "scaled time" parameter

$$t=t(i):=i/n^2. \tag{7}$$

At each step i we color three edges, so the total number of colored edges at that step is $3i=3n^2t$. Heuristically, the probability that an edge is colored is

$$\frac{3n^2t}{\binom{n}{2}}\approx 6t.$$

Thus, in particular we predict that in many ways the uncolored graph should resemble $G(n,p)$ with

$$p=p(t):=1-6t. \tag{8}$$

We would also like a heuristic for the probability that some vertex u has been hit by a color $k\notin S_u$. In this process, u should be getting hit by colors at about the same rate throughout the process. In fact, the proportion of colors in $\textsc{col}\setminus S_u$ that have hit u should be about the same as the proportion $1-p$ of edges in the graph we have colored. Thus, we heuristically assume that the probability $k\notin S_u$ has hit u is $1-p$.

Recall that

$$s=\frac{\frac{\varepsilon}{2}}{\frac{5}{6}+\frac{\varepsilon}{2}}$$

is the probability that (for some fixed color k and vertex u) k is chosen to be in S_u. Note that the expected number of colors in $|\textsc{col}\setminus S_u|$ is $(1-s)|\textsc{col}|=\frac{5}{6}n$ and so

$$|\textsc{col}|=\frac{5}{6}n\cdot\frac{1}{1-s}.$$

We will need to pay careful attention to alternating paths to analyze our process. Heuristically, for some uncolored edge e and a color k, we will assume that there is some function $r(t)$ which we treat as the probability there is no (uv,k)-alternating path at time t. We will guess the appropriate function $r(t)$ using a Poisson heuristic. For a Poisson random variable X, if $\lambda=E[X]$ then $\mathbb{P}(X=0)=e^{-\lambda}$. If we let X be the number of (uv,k)-alternating path at time t, then we ought to have

$$E[X]\approx n^2\cdot(1-p)^3\cdot\left(\frac{1}{|\textsc{col}|}\right)^2,$$

since we have about n^2 choices for possible vertices x and y in $u-x-y-v$, each of the edges ux, xy and yv are colored with probability $1-p$, xy has the color k with probability $1/|\textsc{col}|$, and the edges ux and yv have the same color with probability $1/|\textsc{col}|$ as well. Now substituting the value of $|\textsc{col}|$ gives

$$E[X]\approx n^2\cdot(1-p)^3\cdot\left(\frac{1}{\frac{5}{6}n\cdot\frac{1}{1-s}}\right)^2=\frac{36}{25}(1-s)^2(1-p)^3=\frac{7776}{25}(1-s)^2t^3.$$

Consequently, we heuristically guess that

$$r(t) = \exp\left\{-\frac{7776}{25}(1-s)^2 t^3\right\}.$$

Note that for all $t \leq 1/6$ we have

$$r(t) \geq r(1/6) = \exp\left\{-\frac{36}{25}(1-s)^2\right\} \geq \exp\left\{-\frac{36}{25}\right\} > \frac{1}{5}.$$

3.2. **Variables.** In this subsection, we introduce our family of variables. We start with the variables we intend to track, meaning that we will show that a.a.s. each of these variables stays within a relatively small interval centered around its trajectory. Formally we will use many random variables that are actually sets (not numbers), and when we say we "track" them we mean that we track their cardinalities. We will often abuse notation and omit absolute value signs for the cardinality of sets, i.e. we write S to denote either the set S or its cardinality. In context there should be no confusion. At the end of this subsection we will define a few more variables for which we will obtain only crude upper bounds.

Roughly speaking, the differential equations method is a way to formally argue that a.a.s. certain conditions (bounds on random variables) are maintained as the process runs. Often the goal is to argue that the process does not fail until almost all edges are colored. Thus, our choice of random variables will be motivated by what the process needs to keep going. In our case, the process needs two things: first it needs to be able to choose an uncolored triangle (i.e. the process does not terminate at Substep 4), and then it needs to have some choice of colors for that triangle that obey our coloring rules (i.e. the process does not terminate at Substep 5). Thus, our family of random variables will include one counting the number of uncolored triangles (see the variable Q below), as well as ones counting the number of choices for colors we have for each such triangle (see variables $C^{(1)}, C^{(2)}$). For the differential equation method to work we will need a "closed system" of variables, meaning that if we condition on the current state of the process then the expected one-step change of any variable in our family can be (approximately) written in terms of variables in our family. Thus, our family will have to include several other variables.

We start with the variables used by Bohman, Frieze and Lubetzky [5] for the triangle removal process. This includes Q which is clearly important, as well as another kind of variable which is necessary to make a closed system with Q.

Definition 4. *Let $Q = Q(i)$ be the set of triangles where all three edges are uncolored at step i. For each u, u' we let $Y_{uu'} = Y_{uu'}(i)$ be the set of vertices u'' such that both uu'' and $u'u''$ are uncolored.*

Recalling (7) and (8), the natural heuristic guess for the trajectories (also proved formally in [5]) is

$$Q(i) \approx \binom{n}{3}p^3 \approx \frac{1}{6}n^3p^3 = n^3q(t) \qquad \text{and} \qquad Y_{uu'}(i) \approx np^2 = ny(t),$$

where

$$q(t) := \frac{1}{6}p^3 \qquad \text{and} \qquad y(t) := p^2.$$

We will call these functions $q(t), y(t)$ (i.e. trajectories with the power of n removed) *scaled trajectories*. Before moving to the variables that count color choices, we briefly explain how Q and the Y variables form a closed system.

Let $(\mathscr{F}_i)_{i\ge 0}$ be the "natural filtration" of the process. In particular, conditioning on $\mathscr{F}_i$ tells us exactly what our partial coloring looks like at step i. More formally, our probability space consists of all possible maximal sequences of steps (specifying at each step which triangle, orientation, and colors are chosen), and the partition $\mathscr{F}_i$ groups these sequences according to what happens on the first i steps. The work of Bohman, Frieze and Lubetzky [5] implies that

$$E[\Delta Q(i)|\mathscr{F}_i] = -\sum_{uu'\in E(i)} \frac{Y_{uu'}(i)^2}{Q(i)} + O(1)$$

(where $E(i)$ is the set of uncolored edges at step i) and

$$E[\Delta Y_{uu'}(i)|\mathscr{F}_i] = -\sum_{u''\in Y_{uu'}(i)} \frac{Y_{uu''}(i)+Y_{u'u''}(i)+O(1)}{Q(i)}.$$

Since the conditional expected one-step change of Q and any of the Y variables can be approximately written using only the variables Q and Y, we have a closed system. However, for our coloring process we need several more variables that count color choices. We will now extend our family to include not only variables of types $C^{(1)}_{uu'u''}, C^{(2)}_{uu'u''}$ (which count choices of colors given a fixed oriented triangle), but also several more variables needed to make the system closed again. We will verify later that this system is indeed closed.

The variables of types A through F will all count triples (u,u',u'') and pairs (k,k') that are available at (u,u',u''). For each of these variables we fix some set of vertices and/or colors and count extensions of the fixed set. To illustrate the substructures that these variables count, we will include diagrams that use the following conventions. Closed circles represent vertices that vary (based on some constraint), and open squares represent fixed vertices. Dashed, colored edges represent uncolored edges that have that color available at that edge; a dashed, black edge is a general uncolored edge; and a solid, colored edge is an edge with that color. For example, Figure 4 would indicate that we are fixing u,u',u'' and counting pairs k,k' such that k is available at uu' and uu'' and k' is available at $u'u''$. First we define the type A variables,

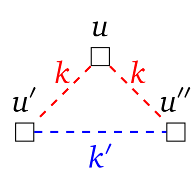


FIGURE 4. A demonstration of the diagram conventions used in this section.

where u', u'' and k' are fixed.

Definition 5. *For each edge $u'u''$ and each color $k'\notin S_{u'}\cup S_{u''}$ we define the random variable $A_{u'u'',k'} = A_{u'u'',k'}(i)$ to be the set of pairs (u,k) such that k is available at uu' and uu'', and $k'\in S_u$.*

Note that technically our definition above does not assume that k' is available at $u'u''$. However whenever that happens to be the case we have for all $(u,k)\in A_{u'u'',k'}$ that the color pair (k,k') is available at the oriented triangle (u,u',u''). Based on our heuristics we predict the following trajectory of $A_{u'u'',k'}$. First, we choose a vertex u with two uncolored edges uu' and uu'' having about np^2 choices. We need to make sure that $k'\in S_u$, which happens with probability s. The number of possible choices for $k\notin S_u\cup S_{u'}\cup S_{u''}$ is $|\text{COL}|(1-s)^3$ and the probability

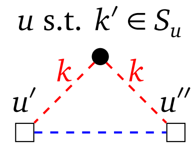


FIGURE 5. A depiction of the $(u,k) \in A_{u'u'',k'}$.

that k did not hit u, u' or u'' in the previous steps is p^3. Finally, with probability r^2 we avoid (uu',k)- and (uu'',k)-alternating paths. Thus,

$$A_{u'u'',k'} \approx |\textsc{col}|ns(1-s)^3p^5r^2 = \frac{5}{6}n^2s(1-s)^2p^5r^2 = n^2a(t),$$

where we define the scaled trajectory

$$a(t) := \frac{5}{6}s(1-s)^2p^5r^2. \tag{9}$$

Next we define the type B variables, which also fix two vertices and a color. For these variables we fix u, u' and k. These are similar to the type A variables but necessary due to the different roles the vertices and colors play in the process.

Definition 6. *For each edge uu' and each color $k \notin S_u \cup S_{u'}$ we define the random variable $B_{uu',k} = B_{uu',k}(i)$ to be the set of pairs (u'',k') such that k is available at uu'', k' is available at and $u'u''$, and $k \in S''_u$.*

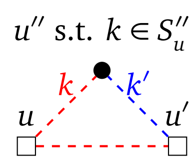


FIGURE 6. A depiction of the $(u'',k') \in B_{uu',k}$.

We heuristically predict that these have the same trajectory as the type A variables. Indeed, the number of possible choices for u'' with uncolored uu'' and $u'u''$ is about np^2. The number of possible choices for k' with $k' \in S_u$ and $k' \notin S_{u'} \cup S_{u''}$ is $|\textsc{col}|s(1-s)^2$ and the probability that $k \notin S_{u''}$ is $(1-s)$. Furthermore, the probability that color k' did not hit neither u' nor u'' is p^2 and the probability that k did not hit u'' is p. Avoiding alternating paths (uu'',k) and $(u'u'',k')$ is again a probability of r^2. Hence,

$$B_{uu',k} \approx |\textsc{col}|ns(1-s)^3p^5r^2 = \frac{5}{6}n^2s(1-s)^2p^5r^2 = n^2b(t),$$

where the scaled trajectory is

$$b(t) := \frac{5}{6}s(1-s)^2p^5r^2 = a(t).$$

Next we define type $C^{(1)}_{uu'u''}$, $C^{(2)}_{uu'u''}$ and $C_{uu'u''}$ variables which fix all of the vertices u,u',u'' and only count colors.

Definition 7. *For each ordered triple (u,u',u'') of uncolored edges we define the random variable $C^{(1)}_{uu'u''} = C^{(1)}_{uu'u''}(i)$ to be the set of colors k' such that k' is 1-available at (u,u',u'') at step i. We define the random variable $C^{(2)}_{uu'u''} = C^{(2)}_{uu'u''}(i)$ to be the set of colors k such that k is 2-available at (u,u',u'') at step i. We also define $C_{uu'u''}(i)$ to be the set of pairs (k,k') available at (u,u',u''). In other words $C_{uu'u''}(i)$ is the Cartesian product $C^{(1)}_{uu'u''} \times C^{(2)}_{uu'u''}$.*

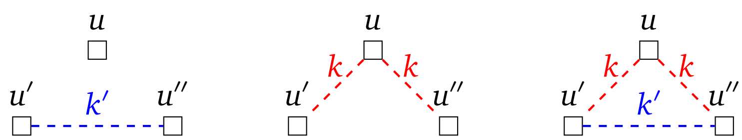


FIGURE 7. Depictions of the $k' \in C^{(1)}_{uu'u''}$, $k \in C^{(2)}_{uu'u''}$, and $(k,k') \in C_{uu'u''}$.

Similarly, as for the previous variables we heuristically predict that

$$C^{(1)}_{uu'u''} \approx |\mathrm{COL}|s(1-s)^2p^2r = \frac{5}{6}ns(1-s)p^2r = nc_1(t),$$

$$C^{(2)}_{uu'u''} \approx |\mathrm{COL}|(1-s)^3p^3r^2 = \frac{5}{6}n(1-s)^2p^3r^2 = nc_2(t),$$

and

$$C_{uu'u''}(i) \approx \frac{25}{36}n^2s(1-s)^3p^5r^3 = n^2c(t),$$

where the scaled trajectories are

$$c_1(t) := \frac{5}{6}s(1-s)p^2r, \quad c_2(t) := \frac{5}{6}(1-s)^2p^3r^2 \quad \text{and} \quad c(t) := \frac{25}{36}s(1-s)^3p^5r^3 = c_1(t)c_2(t).$$

Now we have type D, E and F variables, where one vertex and one color are fixed.

Definition 8. *For each vertex u and each color $k \notin S_u$ we define the random variable $D_{u,k} = D_{u,k}(i)$ to be the set of triples (u',u'',k') such that (k,k') is available at (u,u',u'') at step i.*

Definition 9. *For each vertex u'' and each color $k \notin S_{u''}$ we define the random variable $E_{u'',k} = E_{u'',k}(i)$ to be the set of triples (u,u',k') such that (k,k') is available at (u,u',u'') at step i.*

Definition 10. *For each vertex u'' and each color $k' \notin S_{u''}$ we define the random variable $F_{u'',k'} = F_{u'',k'}(i)$ to be the set of triples (u,u',k) such that (k,k') is available at (u,u',u'') at step i.*

Based on our heuristics we predict the following trajectories. Here, for example, we explain how to obtain the predicted trajectory of $F_{u'',k'}$. First we choose an ordered pair u and u' with all uncolored edges. This gives us about n^2p^3 choices. Next we choose a color k such that $k \notin S_u \cup S_{u'} \cup S_{u''}$ yielding $|\mathrm{COL}|(1-s)^3$ possibilities. Now we observe that the probability that $k' \in S_u$ and $k' \notin S_{u'}$ is $s(1-s)$. Furthermore, the probability that u, u' and u'' are not hit by k is p^3, and the probability that k' did not hit u' is p. Finally, the probability of avoiding alternating paths (uu',k), (uu'',k) and $(u'u'',k')$ is r^3. Thus, $F_{u'',k'} \approx |\mathrm{COL}|n^2s(1-s)^4p^7r^3$.

Trajectories of $D_{u,k}$ and $E_{u'',k}$ can be derived in a similar fashion. Consequently,

$$D_{u,k}, E_{u'',k}, F_{u'',k'} \approx |\mathrm{COL}|n^2s(1-s)^4p^7r^3 = \frac{5}{6}n^3s(1-s)^3p^7r^3$$

with the scaled trajectories

$$d(t), e(t), f(t) := \frac{5}{6}s(1-s)^3p^7r^3.$$

Finally we define our type Z variables, which are useful for tracking which colors become forbidden due to alternating 4-cycles. In particular, for a fixed edge uv and a color k, we keep track of substructures that could eventually cause k to be forbidden at uv due to a potential alternating 4-cycle.

Definition 11. *Fix two vertices u, v, a color $k \notin S_u \cup S_v$ and a vector $(a_1, a_2, a_3) \in \{0,1\}^3$ with $(a_1, a_2, a_3) \neq (1,1,1)$. We define the random variable $Z_{uv,k,a_1,a_2,a_3} = Z_{uv,k,a_1,a_2,a_3}(i)$ to be the number of triples (x, y, k') where x, y are vertices and k' is a color satisfying the following condition. Letting $e_1 := ux$, $e_2 := xy$, $e_3 := yv$, and $k_1 := k'$, $k_2 := k, k_3 := k'$, we have for each $1 \le j \le 3$ that*

- *if $a_j = 0$ then k_j is available at e_j, and*
- *if $a_j = 1$ then e_j is assigned the color k_j.*

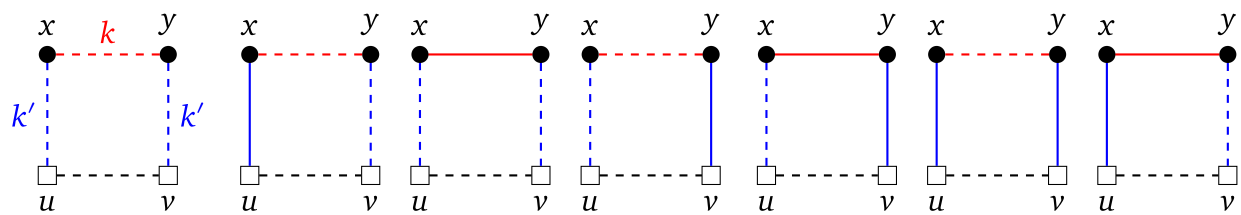


FIGURE 8. Depictions of the $(x, y, k') \in Z_{uv,k,a_1,a_2,a_3}$ for $(a_1, a_2, a_3) \in \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (0,1,1), (1,1,0)\}$ (respectively).

We anticipate the following trajectories. For example, we explain in detail how to predict $Z_{uv,k,0,1,1}$. First we choose an ordered pair x and y such that xy and yv are already colored. For this we should have $n^2 p(1-p)^2$ choices. Next we need to make sure that the color of xy is k. This should happen with probability $1/|\text{COL}|$. The color k' is already determined by the color of yv and k' must be available at ux. In particular, k' must not be in S_u or S_x, which happens with probability $(1-s)^2$. Also k' must not have already hit u or x before, which occurs with probability p^2. Finally there must not be an alternating (ux, k')-path, which happens with probability r. Thus, $Z_{uv,k,0,1,1} \approx n^2 p(1-p)^2 \cdot \frac{1}{|\text{COL}|} \cdot (1-s)^2 \cdot p^2 \cdot r$.

The remaining trajectories can be obtained similarly.

$$Z_{uv,k,0,0,0} \approx |\text{COL}| n^2 (1-s)^6 p^9 r^3 = \frac{5}{6} n^3 (1-s)^5 p^9 r^3,$$
$$Z_{uv,k,1,0,0} \approx Z_{uv,k,0,1,0} \approx Z_{uv,k,0,0,1} \approx n^2 (1-s)^4 (1-p) p^6 r^2,$$
$$Z_{uv,k,1,1,0} \approx Z_{uv,k,1,0,1} \approx Z_{uv,k,0,1,1} \approx \frac{n^2}{|\text{COL}|} (1-s)^2 (1-p)^2 p^3 r = \frac{6}{5} n (1-s)^3 (1-p)^2 p^3 r.$$

Thus we define the following scaled trajectories:

$$z_0(t) := \frac{5}{6}(1-s)^5 p^9 r^3, \quad z_1(t) := (1-s)^4 (1-p) p^6 r^2 \quad \text{and} \quad z_2(t) := \frac{6}{5}(1-s)^3 (1-p)^2 p^3 r.$$

3.3. **Derivatives of the trajectories.** First, we collect all the scaled trajectories:

$$y(t) = p^2,$$

$$\begin{aligned}
q(t) &= \frac{1}{6}p^3,\\
a(t)=b(t) &= \frac{5}{6}s(1-s)^2p^5r^2,\\
c_1(t) &= \frac{5}{6}s(1-s)p^2r,\\
c_2(t) &= \frac{5}{6}(1-s)^2p^3r^2,\\
d(t)=e(t)=f(t) &= \frac{5}{6}s(1-s)^3p^7r^3,\\
z_0(t) &= \frac{5}{6}(1-s)^5p^9r^3,\\
z_1(t) &= (1-s)^4(1-p)p^6r^2,\\
z_2(t) &= \frac{6}{5}(1-s)^3(1-p)^2p^3r.
\end{aligned}$$

These functions satisfy the following system of differential equations. Each differential equation in the system naturally arises from estimating the expected one-step change in one of our random variables. The fact that our scaled trajectories satisfy this system is crucial to our analysis and will be used in our calculations. It is not hard to check (with, for example, a software such as Maple) that the system is satisfied using that $p'(t)=-6$ and $r'(t)=-\frac{648}{25}(1-s)^2(1-p)^2r$. We have:

$$a'(t)=b'(t)=-\frac{5ad}{2qc}-\frac{6a^2z_2}{qc}-\frac{2ay}{q}, \tag{10}$$

$$c_1'(t)=-\frac{5dc_1}{3qc}-\frac{3az_2c_1}{qc}, \tag{11}$$

$$c_2'(t)=-\frac{5dc_2}{2qc}-\frac{6az_2c_2}{qc}, \tag{12}$$

$$d'(t)=e'(t)=f'(t)=-\frac{20d^2}{6qc}-\frac{9az_2d}{qc}-\frac{3yd}{q}, \tag{13}$$

$$z_0'(t)=-\frac{5dz_0}{qc}-\frac{9az_2z_0}{qc}-\frac{3yz_0}{q}, \tag{14}$$

$$z_1'(t)=\frac{az_0}{qc}-\frac{10dz_1}{3qc}-\frac{6az_2z_1}{qc}-\frac{2yz_1}{q}, \tag{15}$$

$$z_2'(t)=\frac{2az_1}{qc}-\frac{5dz_2}{3qc}-\frac{3az_2^2}{qc}-\frac{yz_2}{q}. \tag{16}$$

We will also need a crude bound on the first and second derivatives of the scaled trajectories. Note that all these functions ($a, b,$ etc.) have the form $h_1(t)\exp(h_2(t))$ where h_1 and h_2 are polynomials. It is easy to see that the derivative (and second derivative) of any such function has the form $h_3(t)\exp(h_2(t))$ where $h_3(t)$ is a polynomial. In particular, the first and second derivatives are all $O(1)$ for all $0\le t\le 1$. Thus we have:

Proposition 1. *The first and second derivatives of all the scaled trajectory functions are $O(1)$.*

3.4. **Untracked variables.** In addition to the random variables we already mentioned, which we will track, we will also need several random variables for which we will will establish some necessary, but less precise, bounds in our analysis. In particular, when we consider the maximum one-step change in the Z type variables, we could potentially lose a catastrophic number of triples through alternating paths forbidding edges in two types of pathological substructures.

Definition 12. *(i) Fix two vertices u, v and a color k. We define the random variable $\Xi_{u,v,k} = \Xi_{u,v,k}(i)$ to be the number of pairs (x, y) such that ux has the same color as vy, and xy has the color k. In other words, $\Xi_{u,v,k}$ is the number of alternating (uv, k)-paths. See Figure 9a.*

(ii) Fix four vertices u, u', v, v' and a color k. We define the random variable $\Phi_{u,u',v,v'} = \Phi_{u,u',v,v'}(i)$ to be the number of pairs (x, y) such that ux has the same color as $u'y$ and vx has the same color as $v'y$. See Figure 9b.

(iii) Fix two vertices u, u'' and colors k, k''. We define the random variable $\Psi_{u,u'',k,k''} = \Psi_{u,u'',k,k''}(i)$ to be the number of triples (x, y, z) such that ux has the same color as zu'', xy has the color k and yz has the color k''. See Figure 9c.

(iv) Fix three vertices u, v, w. We define the random variable $\Lambda_{u,v,w} = \Lambda_{u,v,w}(i)$ to be the number of pairs (x, y) such that ux has the same color as vy, and vx has the same color as wy. See Figure 9d.

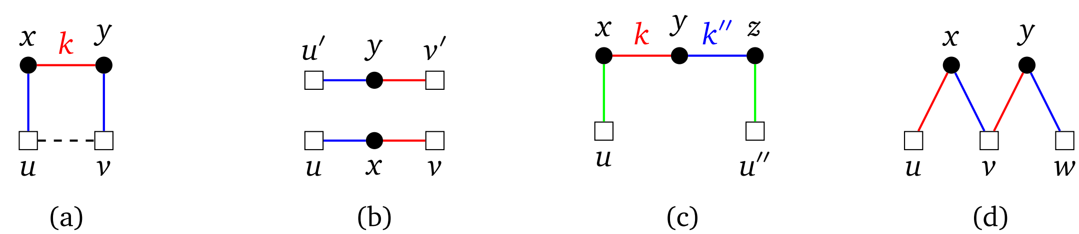


FIGURE 9. Depictions of the $(x, y) \in \Xi_{u,v,k}$, $(x, y) \in \Phi_{u,u',v,v'}$, $(x, y, z) \in \Psi_{u,u'',k,k''}$ and $(x, y) \in \Lambda_{u,v,w}$.

4. THE GOOD EVENT

In this section we define the good event $\mathcal{E}_i$, which among other things stipulates that every uncolored triangle (u, u', u'') still has plenty of available pairs of colors (k, k'). More specifically, $\mathcal{E}_i$ will stipulate that all of our tracked variables are within a small window of their respective trajectories we derived in Section 3. The event $\mathcal{E}_i$ will also stipulate some crude upper bounds on certain other variables. Note that in the process, we choose ε, which gives us $s(\varepsilon)$. Then we let

$$\delta := 10^{-7} s(1-s)^4$$

and define below all of the error functions g_q, g_y, etc..

For any step i' we let $t' = t(i')$. We formally define the *good event* $\mathcal{E}_i$ to be the event that for all $i' \le i$ we have the following conditions (below, functions are evaluated at $i = i'$, $t = t'$):

(I) we have
$$\left|Q - n^3 q(t)\right| \le n^3 g_q,$$

(II) for each uncolored edge uu' we have
$$|Y_{uu'} - n y(t)| \le n g_y,$$

(III) for each uncolored edge $u'u''$ we have

$$\left|A_{u'u'',k'}-n^2a(t)\right|\le n^2g_{ab},$$

(IV) for each uncolored edge uu' and color k we have

$$\left|B_{uu',k}-n^2b(t)\right|\le n^2g_{ab},$$

(V) for each triple (u,u',u'') of uncolored edges we have

$$\left|C^{(1)}_{uu'u''}-nc_1(t)\right|\le ng_{c1},$$

(VI) for each triple (u,u',u'') of uncolored edges we have

$$\left|C^{(2)}_{uu'u''}-nc_2(t)\right|\le ng_{c2},$$

(VII) for each vertex u and color k available at u we have

$$\left|D_{u,k}-n^3d(t)\right|\le n^3g_{def},$$

(VIII) for each vertex u'' and color k available at u'' we have

$$\left|E_{u'',k}-n^3e(t)\right|\le n^3g_{def},$$

(IX) for each vertex u'' and color k' available at u'' we have

$$\left|F_{u'',k'}-n^3f(t)\right|\le n^3g_{def},$$

(X) for each uncolored edge uv and color k we have

$$\left|Z_{uv,k,0,0,0}-n^3z_0(t)\right|\le n^3g_0,$$

(XI) for each uncolored edge uv and color k we have

$$\left|Z_{uv,k,1,0,0}-n^2z_1(t)\right|\le n^2g_1,$$

$$\left|Z_{uv,k,0,1,0}-n^2z_1(t)\right|\le n^2g_1,$$

$$\left|Z_{uv,k,0,0,1}-n^2z_1(t)\right|\le n^2g_1,$$

(XII) for each uncolored edge uv and color k we have

$$\left|Z_{uv,k,1,1,0}-nz_2(t)\right|\le ng_2,$$

$$\left|Z_{uv,k,1,0,1}-nz_2(t)\right|\le ng_2,$$

$$\left|Z_{uv,k,0,1,1}-nz_2(t)\right|\le ng_2.$$

(XIII) for all u,v,k we have

$$\Xi_{u,v,k}\le n^{4\delta},$$

for all u,u',v,v' we have

$$\Phi_{u,u',v,v'}\le n^{4\delta},$$

for all u,u'',k,k' we have

$$\Psi_{u,u'',k,k''}\le n^{4\delta},$$

and for all u,v,w we have

$$\Lambda_{u,v,w}\le n^{4\delta}.$$

Recall that we define the random variable $C_{uu'u''} = C_{uu'u''}(i)$ as $C^{(1)}_{uu'u''} \times C^{(2)}_{uu'u''}$. This will count the number of pairs (k, k') that are available at (u, u', u'') at step i. In addition, we let

$$c(t) := c_1(t)c_2(t) \quad \text{and} \quad g_c := 2(c_2 g_{c_1} + c_1 g_{c_2}).$$

Since $g_{c_1} = o(c_1)$, we get in the good event,

$$C_{uu'u''} \le n(c_1 + g_{c_1}) \cdot n(c_2 + g_{c_2}) = n^2(c + c_1 g_{c_2} + g_{c_1} c_2 + g_{c_1} g_{c_2}) \le n^2(c + g_c)$$

and similarly $C_{uu'u''} \ge n^2(c - g_c)$. Thus,

$$\left| C_{uu'u''} - n^2 c(t) \right| \le n^2 g_c.$$

We let

$$i_{max} := \frac{1}{6} n^2 \left(1 - n^{-\delta}\right), \qquad t_{max} := \frac{i_{max}}{n^2} = \frac{1}{6}\left(1 - n^{-\delta}\right),$$

and note that

$$p(t_{max}) = 1 - 6 t_{max} = n^{-\delta}.$$

Let

$$\kappa = \kappa(s) = 10000 s^{-1}(1-s)^{-4} \qquad \text{and} \qquad \omega = 100(\kappa + 1)\delta.$$

Note that $\omega = 100\delta + 1/10 = 10^{-5} s(1-s)^4 + 1/10 < 1/4$, since $s = s(\varepsilon)$ and $\varepsilon < 1/100$. We define the error functions as follows:

$$\begin{aligned}
g_y(t) &= n^{-1/2+\delta},\\
g_q(t) &= n^{-1+2\delta},\\
g_{ab}(t) &= n^{-\omega} p(t)^{-100\kappa},\\
g_{c_1}(t) &= n^{-\omega} p(t)^{-100\kappa-2},\\
g_{c_2}(t) &= n^{-\omega} p(t)^{-100\kappa-1},\\
g_{def}(t) &= n^{-\omega} p(t)^{-100\kappa+3},\\
g_0(t) &= n^{-\omega} p(t)^{-100\kappa+5},\\
g_1(t) &= n^{-\omega} p(t)^{-100\kappa+1},\\
g_2(t) &= n^{-\omega} p(t)^{-100\kappa-1}.
\end{aligned}$$

Note that

$$\frac{g_q(t)}{q(t)} = \frac{n^{-1+2\delta}}{\frac{1}{6} p(t)^3} \le 6 n^{-1+5\delta} = o(1)$$

since $p \ge n^{-\delta}$ and $\delta < 1/1000$. Furthermore, using these and that $\varepsilon/2 \le s \le 3\varepsilon/5$ and $r \ge 1/6$, we obtain

$$\frac{g_c}{c} = \frac{2(c_2 g_{c_1} + c_1 g_{c_2})}{c} = O\left(\frac{p^3 \cdot n^{-\omega} p(t)^{-100\kappa-2} + p^2 \cdot n^{-\omega} p(t)^{-100\kappa-1}}{p^5} \right) = O\left(n^{-\omega + (100\kappa - 4)\delta}\right) = o(1).$$

It is also routine to check that the error function satisfy the following, which will be required at a crucial point in our analysis:

$$\begin{aligned}
& g'_{ab} - 30\kappa\left(p^2 g_2 + p^{-1} g_{ab} + p^{-1} g_c\right) \qquad (17)\\
& \qquad = n^{-\omega}\left(570 p^{-100\kappa-1} - 30 p^{-100\kappa+1} - 120 p^{-100\kappa}\right) = \Omega(1),\\
& g'_{c_1} - 30\kappa(p^{-1} g_2 + p^{-3} g_{ab} + p^{-4} g_c + p^{-6} g_{def}) \qquad (18)
\end{aligned}$$

$$= n^{-\omega}\left((420\kappa + 12)p^{-100\kappa-3} - 30\kappa p^{-100\kappa-2}\right) = \Omega(1),$$

$$g'_{c_2} - 30\kappa(p^{-1}g_2 + p^{-2}g_{ab} + p^{-3}g_c + p^{-5}g_{def}) \tag{19}$$
$$= n^{-\omega}(390\kappa + 6)p^{-100\kappa-2} = \Omega(1),$$

$$g'_{def} - 30\kappa\left(p^4 g_2 + p^{-1}g_{def} + p^2 g_{ab} + p g_c\right) \tag{20}$$
$$= n^{-\omega}\left((420\kappa - 18)p^{-100\kappa+2} - 30\kappa p^{-100\kappa+3}\right) = \Omega(1),$$

$$g'_0 - 30\kappa\left(p^6 g_2 + p g_{def} + p^4 g_{ab} + p^3 g_c\right) \tag{21}$$
$$= n^{-\omega}\left((420\kappa - 30)p^{-100\kappa+4} - 30\kappa p^{-100\kappa+5}\right) = \Omega(1),$$

$$g'_1 - 40\kappa\left(p^3 g_2 + p^{-2}g_{def} + p g_{ab} + p^{-1}g_c\right) \tag{22}$$
$$= n^{-\omega}\left((440\kappa - 6)p^{-100\kappa} - 80\kappa p^{-100\kappa+1} - 40p^{-100\kappa+2}\kappa\right) = \Omega(1),$$

$$g'_2 - 40\kappa\left(p^{-1}g_2 + p^{-5}g_{def} + p^{-2}g_{ab} + p^{-3}g_c\right) \tag{23}$$
$$= n^{-\omega}(320\kappa + 6)p^{-100\kappa-2} = \Omega(1).$$

Finally note that all error functions have the form $n^{-\omega}p^{-h}$ where $h \geq -100\kappa - 2$ is a constant. So the first derivative is a constant times $n^{-\omega}p^{-h-1}$ and the second derivative is a constant times $n^{-\omega}p^{-h-2}$. In particular for all $0 \leq t \leq 1$ the second derivative of any error function is

$$O\left(n^{-\omega}p^{-100\kappa-4}\right) = O\left(n^{-\omega+(100\kappa+4)\delta}\right)$$

and similarly for the first derivative. Thus we have:

Proposition 2. *The first and second derivatives of all the error functions are $O(1)$.*

5. Some helpful bounds that hold in the good event

Some of these bounds will be sharp and others will be more crude upper bounds.

5.1. **Sharp estimates.** In order to track certain variables we will need to estimate the probabilities of the following events.

- For a color k^* available at v at step i, we let $\mathscr{L}_{v,k^*} = \mathscr{L}_{v,k^*}(i)$ be the event that v gets hit by k at step i.
- For a color k^* available at edge vw, we let $\mathscr{M}_{vw,k^*} = \mathscr{M}_{vw,k^*}(i)$ be the event that vw becomes the color k^* at step i.
- For an uncolored edge vw, we let $\mathscr{M}_{vw,\bullet} = \mathscr{M}_{vw,\bullet}(i)$ be the event that vw gets colored any color at step i.
- For any color k and an uncolored edge uv, let $\mathscr{N}_{uv,k} = \mathscr{N}_{uv,k}(i)$ be the event that the edge uv will become part of a (uv,k)-alternating path.

Claim 1. *Assuming the good event $\mathscr{E}_i$ holds, we have:*

(i) $n^{-2}\left[\frac{5d}{6qc} - \kappa\left(p^{-8}g_{def} + p^{-6}g_c\right)\right] \leq \mathbb{P}(\mathscr{L}_{v,k^*}) \leq n^{-2}\left[\frac{5d}{6qc} + \kappa\left(p^{-8}g_{def} + p^{-6}g_c\right)\right]$,

(ii) $n^{-3}\left[\frac{a}{qc} - \kappa p^{-8}\left(g_{ab} + g_c\right)\right] \leq \mathbb{P}(\mathscr{M}_{vw,k^*}) \leq n^{-3}\left[\frac{a}{qc} + \kappa p^{-8}\left(g_{ab} + g_c\right)\right]$,

(iii) $n^{-2}\frac{y}{q} - O(n^{-5/2+4\delta}) \leq \mathbb{P}(\mathscr{M}_{vw,\bullet}) \leq n^{-2}\frac{y}{q} + O(n^{-5/2+4\delta})$, *and*

(iv) $n^{-2}\left[\frac{3az_2}{qc} - \kappa\left(p^{-3}g_2 + p^{-5}g_{ab} + p^{-5}g_c\right)\right] \leq \mathbb{P}(\mathscr{N}_{uv,k}) \leq n^{-2}\left[\frac{3az_2}{qc} + \kappa\left(p^{-3}g_2 + p^{-5}g_{ab} + p^{-5}g_c\right)\right]$.

We also state some simple bounds related to the above probabilities. These bounds easily follow from the trajectories given in Section 3, $0 \le p \le 1$, $1/5 \le r \le 1$ and the assumption that $s > 0$ is sufficiently small. Therefore, the proof is omitted.

Claim 2. *We have:*

$$\frac{d}{qc} \le 50p^{-1}, \qquad \frac{az_2}{qc} \le 10, \qquad \frac{a}{qc} \le 10p^{-3}, \qquad \text{and} \qquad \frac{y}{q} \le 10p^{-1}.$$

We will use the following upper and lower bounds throughout the proof of Claim 1. The following fact is easily checked, and thus we omit the proof.

Fact 1. *Let $x = x(n), y = y(n), z = z(n)$ with $x, y, z \in (0,1)$ and $y, z, = o(1)$. Then, for sufficiently large n, we have*

$$\frac{1+x}{(1-y)(1-z)} \le 1+2x+2y+2z \qquad \text{and} \qquad \frac{1-x}{(1+y)(1+z)} \ge 1-2x-2y-2z.$$

Proof of Claim 1. We prove each statement separately.

Part (i): By using Fact 1 we get

$$\begin{aligned}\mathbb{P}(\mathscr{L}_{v,k^*}) &= \frac{1}{2}\sum_{(u',u'',k')\in D_{v,k^*}} \frac{1}{3QC_{vu'u''}} + \sum_{(u,u',k')\in E_{v,k^*}} \frac{1}{3QC_{uu'v}} + \sum_{(u,u',k)\in F_{v,k^*}} \frac{1}{3QC_{uu'v}}\\ &\le \frac{5}{2}\cdot\frac{n^3(d+g_{def})}{3n^3\left(q-g_q\right)\cdot n^2\left(c-g_c\right)} = n^{-2}\cdot\frac{5d}{6qc}\cdot\frac{1+\frac{g_{def}}{d}}{\left(1-\frac{g_q}{q}\right)\left(1-\frac{g_c}{c}\right)}.\end{aligned}$$

Since g_q/q and g_c/c are $o(1)$, we use Fact 1 and next $g_q/q = o(g_{def}/d)$ to obtain

$$\begin{aligned}\mathbb{P}(\mathscr{L}_{v,k^*}) &\le n^{-2}\cdot\frac{5d}{6qc}\left(1+\frac{2g_{def}}{d}+\frac{2g_q}{q}+\frac{2g_c}{c}\right) \le n^{-2}\cdot\frac{5d}{6qc}\left(1+\frac{4g_{def}}{d}+\frac{2g_c}{c}\right)\\ &= n^{-2}\left(\frac{5d}{6qc}+\frac{144}{5}\frac{1}{(1-s)^3sr^3}p^{-8}g_{def}+\frac{432}{25}\frac{1}{(1-s)^3sr^3}p^{-6}g_c\right).\end{aligned}$$

Finally, since $r^{-3} \le 5^3$ and $\kappa(s) = 10^4 s^{-1}(1-s)^{-4}$, we obtain the required upper bound on $\mathbb{P}(\mathscr{L}_{v,k^*})$. Using a similar calculation and the lower bound in Fact 1 gives the lower bound.

Part (ii): Using similar calculations as Part (ii) yields

$$\begin{aligned}\mathbb{P}(\mathscr{M}_{vw,k^*}) &= \sum_{(u,k)\in A_{vw,k^*}} \frac{1}{3QC_{uvw}} + \sum_{(u'',k')\in B_{vw,k^*}} \frac{1}{3QC_{vwu''}} + \sum_{(u'',k')\in B_{wv,k^*}} \frac{1}{3QC_{wvu''}}\\ &\le 3\cdot\frac{n^2(a+g_{ab})}{3n^3\left(q-g_q\right)\cdot n^2\left(c-g_c\right)} = n^{-3}\frac{a}{qc}\cdot\frac{1+\frac{g_{ab}}{a}}{\left(1-\frac{g_q}{q}\right)\left(1-\frac{g_c}{c}\right)}\\ &\le n^{-3}\cdot\frac{a}{qc}\left(1+\frac{2g_{ab}}{a}+\frac{2g_q}{q}+\frac{2g_c}{c}\right) \le n^{-3}\cdot\frac{a}{qc}\left(1+\frac{4g_{ab}}{a}+\frac{2g_c}{c}\right)\\ &= n^{-3}\left(\frac{a}{qc}+\frac{864}{25}\frac{1}{(1-s)^3sr^3}p^{-8}g_{ab}+\frac{2592}{125}\frac{1}{(1-s)^4sr^4}p^{-8}g_c\right)\\ &\le n^{-3}\left[\frac{a}{qc}+\kappa p^{-8}\left(g_{ab}+g_c\right)\right],\end{aligned}$$

where for the latter term we use the fact that $r \geq \exp\left\{-\frac{36}{25}\right\}$.

Part (iii): We have

$$\mathbb{P}(\mathcal{M}_{vw,\bullet}) = \frac{Y_{vw}}{Q} \leq \frac{ny+n^{1/2+\delta}}{n^3q-n^{2+2\delta}} = n^{-2}\frac{y}{q}\left(\frac{1+n^{-1/2+\delta}\frac{1}{y}}{1-n^{-1+2\delta}\frac{1}{q}}\right) = n^{-2}\frac{y}{q}\left(\frac{1+n^{-1/2+3\delta}}{1-6n^{-1+5\delta}}\right)$$
$$= n^{-2}\frac{y}{q}\left(1+O(n^{-1/2+3\delta})\right) = n^{-2}\frac{y}{q}+O(n^{-5/2+4\delta}).$$

Part (iv): Finally note that

$$\mathbb{P}(\mathcal{N}_{uv,k}) = \sum_{(x,y,k')\in Z_{uv,k,1,0,1}} \mathbb{P}(\mathcal{M}_{xy,k}) + \sum_{(x,y,k')\in Z_{uv,k,0,1,1}} \mathbb{P}(\mathcal{M}_{ux,k'}) + \sum_{(x,y,k')\in Z_{uv,k,1,1,0}} \mathbb{P}(\mathcal{M}_{vy,k'})$$
$$\leq 3\cdot n\,(z_2+g_2)\cdot n^{-3}\left[\frac{a}{qc}+\kappa\left(p^{-8}g_{ab}+p^{-8}g_c\right)\right]$$
$$\leq n^{-2}\left[\frac{3az_2}{qc}+\kappa\left(p^{-3}g_2+p^{-5}g_{ab}+p^{-5}g_c\right)\right],$$

where in the latter we use the fact that $g_2 \leq 10z_2/11$ (since $g_2 = o(z_2)$) and $z_2 \leq p^3/3$. □

5.2. **Crude upper bounds.** Here we will (crudely) bound probabilities of intersections of the events defined in the previous subsection.

Claim 3. *The following holds in the good event $\mathcal{E}_i$.*

(i) *Fix vertices $v \neq v'$ and colors k and k' (where we allow $k=k'$). Then,*
$$\mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{L}_{v',k'}(i)) = O\left(n^{-3+8\delta}\right).$$

(ii) *Fix a vertex v, colors k and k' and an edge e.*
- *If $k' \neq k$, then*
$$\mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{M}_{e,k'}(i)) = O\left(n^{-4+8\delta}\right).$$
- *If v is not incident with e, then*
$$\mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{M}_{e,k'}(i)) \leq \mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{M}_{e,\bullet}(i)) = O\left(n^{-4+8\delta}\right).$$
- *If v is incident with e, then*
$$\mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{M}_{e,k'}(i)) \leq \mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{M}_{e,\bullet}(i)) = O\left(n^{-3+8\delta}\right).$$

(iii) *Fix distinct (but possibly adjacent) edges e and e' and a color k. Then,*
$$\mathbb{P}(\mathcal{M}_{e,k}(i)\cap\mathcal{M}_{e',\bullet}(i)) = O\left(n^{-4+8\delta}\right) \quad \text{and} \quad \mathbb{P}(\mathcal{M}_{e,\bullet}(i)\cap\mathcal{M}_{e',\bullet}(i)) = O\left(n^{-3+8\delta}\right).$$

(iv) *Fix a vertex v, colors k and k' (possibly equal) and an edge e (possibly incident with v). Then,*
$$\mathbb{P}(\mathcal{L}_{v,k}(i)\cap\mathcal{N}_{e,k'}(i)) = O\left(n^{-3+8\delta}\right).$$

(v) *Fix distinct (but possibly adjacent) edges e,e', and colors k,k' (possibly equal). Then,*
$$\mathbb{P}(\mathcal{N}_{e,k}\cap\mathcal{N}_{e',k'}) = O\left(n^{-3+12\delta}\right).$$

(vi) *Fix edges e,e' and colors k,k'.*

- *If we assume nothing about e, e' being distinct or nonadjacent or k, k' being distinct, then,*
$$\mathbb{P}(\mathcal{N}_{e,k} \cap \mathcal{M}_{e',k'}) \le \mathbb{P}(\mathcal{N}_{e,k} \cap \mathcal{M}_{e',\bullet}) = O\left(n^{-3+8\delta}\right).$$
- *If $k \neq k'$ and $e \neq e'$ are adjacent, then*
$$\mathbb{P}(\mathcal{N}_{e,k} \cap \mathcal{M}_{e',k'}) = O\left(n^{-4+8\delta}\right).$$
- *Suppose $e = uv$ and $e' = xy$ are distinct and nonadjacent, and $k = k'$. Suppose further that it is not the case that ux has the same color as vy or that uy has the same color as vx. Then,*
$$\mathbb{P}(\mathcal{N}_{e,k} \cap \mathcal{M}_{e',k'}) = O\left(n^{-4+8\delta}\right).$$

Proof. The above bounds have fairly straightforward proofs and, therefore, we omit most of them. Here we only show details for bounds in Parts (i) and (iv).

We start with the following observation. Fix any oriented triangle (v, v', v'') and pair of colors (k, k'). The probability that at step i we choose (v, v', v'') to color, and then choose the color pair (k, k') to use, is
$$\frac{1}{3Q} \cdot \frac{1}{C_{vv'v''}} = O\left(\frac{1}{n^3 q \cdot n^2 c}\right) = O\left(\frac{1}{n^3 p^3 \cdot n^2 p^5}\right) = O\left(n^{-5+8\delta}\right). \tag{24}$$

Part (i): For the event $\mathcal{L}_{v,k} \cap \mathcal{L}_{v',k'}$ to happen, the triangle chosen at step i must contain both v and v' and so there are a linear number of choices for the triangle. The color pair must include k so there is at most a linear number of choices for the color pair. Since each possibility occurs with probability at most $O\left(n^{-5+8\delta}\right)$ by (24), we have
$$\mathbb{P}(\mathcal{L}_{v,k}(i) \cap \mathcal{L}_{v',k'}(i)) = O(n) \cdot O(n) \cdot O\left(n^{-5+8\delta}\right) = O\left(n^{-3+8\delta}\right).$$

Part (iv): Suppose e is an uncolored edge at step i. Let $P(e,k) = P(e,k,i)$ be the set of all pairs (e^*, k^*) where e^* is an edge and k^* is a color such that coloring e^* the color k^* would forbid k at e through an alternating 4-path. More precisely, if $e = wx$ then $P(e,k)$ is the following set:
$$\{(yz, k) : yz \text{ is not adjacent to } wx, \text{ and } wy \text{ has the same color as } zx\}$$
$$\cup \{(wy, k'') : \text{ there exists some } z \text{ where } yz \text{ has color } k \text{ and } zx \text{ has color } k''\}$$
$$\cup \{(xz, k'') : \text{ there exists some } y \text{ where } wy \text{ has color } k'' \text{ and } yz \text{ has color } k\}.$$

We split the proof into cases. Due to (24) it suffices to show that the number of choices for the triangle (containing v) and colors (containing k) is at most $O(n^2)$. In order for $\mathcal{N}_{e,k'}$ to happen there must be some pair $(e^*, k^*) \in P(e, k')$ such that e^* gets assigned the color k^*.

Suppose that e^* is adjacent to e and $k^* = k$. Since no vertex is adjacent to more than two edges of the same color, there are $O(1)$ choices for (e^*, k^*) with this property. There are $O(n)$ triangles containing e^*, and $O(n)$ ways to choose the other color in the color pair.

Now suppose that e^* is adjacent to e and $k^* \neq k$. There are $O(n)$ choices for (e^*, k^*) with this property. There are $O(n)$ triangles containing e^*, and the color pair must consist of k and k^*.

Next assume that e^* is not adjacent to e and does not contain v. There are $O(n)$ choices for e^*, and once we choose one the triangle is determined. One color must be k and we have $O(n)$ choices for the other color.

Finally assume that e^* is not adjacent to e and contains v. There are $O(1)$ choices for e^*, and so $O(n)$ choices for the triangle. One color must be k and we have $O(n)$ choices for the other color. This completes the proof of Part (iv).

As we mentioned, the proofs for the rest of the parts of the claim are very similar and the reader can easily check them. Some of them use the bounds in (XIII). □

6. Variables Q and Y

We now begin verifying that the good event holds, starting with (I) and (II). Both of the variables Q and Y were tracked by Bohman, Frieze and Lubetzky in [5], so we will use a weaker form of their results. They showed that a.a.s. for all

$$i \le i_0 = \frac{1}{6}n^2 - \frac{5}{3}n^{7/4}\log^{5/4} n$$

we have

$$n^3 q(t) - n^2 \log n \cdot \frac{(5-30\log p(t))^2}{p(t)} \le Q(i) \le n^3 q(t) + \frac{1}{3}n^2 p(t), \text{ and}$$

$$|Y_{uu'} - y(t)n| \le \sqrt{n\log n}\cdot(5-30\log p(t)) \text{ for all } uu'.$$

These bounds are better than we need so we will loosen and simplify them. Note that as long as $\delta < 1/4$ we have that $i_{max} \le i_0$. Thus, using that $p(t) \ge p(t_{max}) = n^{-\delta}$ the above bounds on Q and Y imply that for all $i \le i_{max}$ that

$$\left|Q - n^3 q(t)\right| \le n^{2+2\delta}$$

and

$$|Y_{uu'} - y(t)n| \le n^{1/2+\delta}.$$

Thus, $\mathcal{E}_{i_{max}}$ a.a.s. does not fail due to conditions (I) or (II).

7. Variable A

In this section we bound the probability that $\mathcal{E}_{i_{max}}$ fails due to a variable of type A straying too far from its trajectory and violating Condition (III). Several of the sections that follow will have a very similar structure, so we will explain our reasoning carefully in this section so we can go faster in future sections. In addition, we will only show details for representatives of the four following groups of variables

$$A_{u'u'',k'} \in \{A_{u'u'',k'}, B_{uu',k}\}, \qquad C^{(1)}_{uu'u''} \in \{C^{(1)}_{uu'u''}, C^{(2)}_{uu'u''}\}, \qquad D_{u,k} \in \{D_{u,k}, E_{u'',k}, F_{u'',k'}\}$$

and

$$Z_{uv,k,0,0,0} \in \{Z_{uv,k,0,0,0}, Z_{uv,k,1,0,0}, Z_{uv,k,0,1,0}, Z_{uv,k,0,0,1}, Z_{uv,k,0,1,1}, Z_{uv,k,1,0,1}, Z_{uv,k,1,1,0}\}.$$

The variables within a group require similar calculations, and in some cases have the exact same trajectory. In the case of the Z variables, extending the work on the remaining types requires some routine, but tedious, additional details which we omit for readability.

Each of Conditions (III)-(XII) states that some random variable lies within some interval centered at its trajectory, i.e. it is equivalent to a statement of the form

$$X(i) \in [x_1(t), x_2(t)],$$

where $X = X(i)$ is our random variable and x_1, x_2 are deterministic functions of t (possibly depending on n as well). We use the following strategy to bound the probability that X leaves the interval. First we define a pair of auxiliary random variables

$$X^+(i) := \begin{cases} X(i) - x_2(t) & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ X^+(i-1) & \text{otherwise,} \end{cases}$$

and

$$X^-(i) := \begin{cases} X(i) - x_1(t) & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ X^-(i-1) & \text{otherwise.} \end{cases}$$

Note that if $\mathscr{E}_{i-1}$ holds but $\mathscr{E}_i$ fails due to X leaving its interval, then we have either $X^+(i) > 0$ or $X^-(i) < 0$. To bound the probability of that event we show that X^+ is a supermartingale, and X^- is a submartingale (showing the latter is typically very similar to the former and we will often show less work here). The bound on the failure probability then follows from Freedman's inequality.

We now proceed to apply the strategy described above to the variables of type A. We let

$$A^{\pm}_{u'u'',k'} = A^{\pm}_{u'u'',k'}(i) := \begin{cases} A_{u'u'',k'} - n^2(a(t) \pm g_{ab}(t)) & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ A^{\pm}_{u'u'',k'}(i-1) & \text{otherwise.} \end{cases}$$

To check that $A^+_{u'u'',k'}$ is a supermartingale, we must show that $\mathbb{E}[\Delta A^+_{u'u'',k'} | \mathscr{F}_i] \le 0$ where we define $\Delta A^+_{u'u'',k'} := A^+_{u'u'',k'}(i+1) - A^+_{u'u'',k'}(i)$. We first deal with a trivial case. If at step i we have that $\mathscr{E}_i$ fails, then by definition we have $\Delta A^+_{u'u'',k'} = 0$ and we are done. Henceforth assume that $\mathscr{E}_i$ holds.

We estimate the one-step change in $A_{u'u'',k'}$. This variable never increases, and each pair $(u,k) \in A_{u'u'',k'}$ can be lost in one of the following ways:

- one of the vertices u, u', u'' can get hit by k,
- one of the edges uu', uu'' can have k forbidden due to a potential alternating 4-cycle, or
- one of the edges uu', uu'' can get colored.

Thus, for each pair $(u,k) \in A_{u'u'',k'}(i)$, the probability that $(u,k) \notin A_{u'u'',k'}(i+1)$ is

$$\mathbb{P}\left[\bigcup_{z \in \{u,u',u''\}} \mathscr{L}_{z,k} \;\cup\; \bigcup_{e \in \{uu',uu''\}} \left(\mathscr{N}_{e,k} \cup \mathscr{M}_{e,\bullet} \right) \right]$$

and so

$$\mathbb{E}[\Delta A_{u'u'',k'} | \mathscr{F}_i] = -\sum_{(u,k) \in A_{u'u'',k'}} \mathbb{P}\left[\bigcup_{z \in \{u,u',u''\}} \mathscr{L}_{z,k} \;\cup\; \bigcup_{e \in \{uu',uu''\}} \left(\mathscr{N}_{e,k} \cup \mathscr{M}_{e,\bullet} \right) \right].$$

Now we will approximate the above probability by using the union bound with an error term as follows. Let $E_1, \ldots, E_k$ be the set of events. Then,

$$\sum_{i=1}^{k} \mathbb{P}(E_i) - \sum_{1 \le i < j \le k} \mathbb{P}(E_i \cap E_j) \le \mathbb{P}\left[\bigcup_{i=1}^{k} E_i \right] \le \sum_{i=1}^{k} \mathbb{P}(E_i). \tag{25}$$

This together with Claim 3 and the assumption that the good event $\mathscr{E}_i$ holds implies that

$$\mathbb{P}\left[\bigcup_{z\in\{u,u',u''\}} \mathscr{L}_{z,k} \cup \bigcup_{e\in\{uu',uu''\}} \left(\mathscr{N}_{e,k}\cup\mathscr{M}_{e,\bullet}\right)\right] = \sum_{z\in\{u,u',u''\}} \mathbb{P}\left(\mathscr{L}_{z,k}\right) + \sum_{e\in\{uu',uu''\}} \left[\mathbb{P}\left(\mathscr{N}_{e,k}\right)+\mathbb{P}\left(\mathscr{M}_{e,\bullet}\right)\right] + O(n^{-3+12\delta}).$$

Consequently,

$$\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_i] = -\sum_{(u,k)\in A_{u'u'',k'}} \left\{ \sum_{z=\{u,u',u''\}} \mathbb{P}\left(\mathscr{L}_{z,k}\right) + \sum_{e=\{uu',uu''\}} \left[\mathbb{P}\left(\mathscr{N}_{e,k}\right)+\mathbb{P}\left(\mathscr{M}_{e,\bullet}\right)\right] + O(n^{-3+12\delta}) \right\}.$$

We will again use the assumption that $\mathscr{E}_i$ holds to give deterministic upper and lower bounds on $\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_i]$. Due Claim 1 we have

$$\begin{aligned}\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_i] \le & -n^2(a-g_{ab})\left\{3n^{-2}\left[\frac{5d}{6qc} - \kappa\left(p^{-8}g_{def}+p^{-6}g_c\right)\right]\right.\\ & \left. +2n^{-2}\left[\frac{3az_2}{qc}+\frac{y}{q}-\kappa\left(p^{-3}g_2+p^{-5}g_{ab}+p^{-5}g_c\right)\right] + O\left(n^{-3+12\delta}+n^{-5/2+4\delta}\right)\right\}\\ \le & -(a-g_{ab})\left\{\frac{5d}{2qc}+\frac{6az_2}{qc}+\frac{2y}{q} - 10\kappa\left(p^{-3}g_2+p^{-5}g_{ab}+p^{-6}g_c\right)\right\} + O(n^{-1/2+4\delta}),\end{aligned}$$

where on the last line we used the fact that $p^{-3}g_{def} = g_{ab}$ and assumed that δ is small to simplify the big-O term. Now we will take the above expression and separate the "main terms" from the "first-order error term" (the terms involving error functions g_{ab}, etc.) and the "lesser-order error terms" (in the big-O). We will be precise for the main terms and generous for error terms. By Claim 2 we get

$$g_{ab}\left(\frac{5d}{2qc}+\frac{6az_2}{qc}+\frac{2y}{q}\right) \le g_{ab}\left(125p^{-1}+60+20p^{-1}\right) \le 205 g_{ab}p^{-1} \le \kappa g_{ab}p^{-1}.$$

Recalling that κ is large, $a(t)\le p^5$ (see (9)), and $g_{ab}(t) = o(a(t))$, we obtain

$$\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_i] \le -\frac{5ad}{2qc} - \frac{6a^2z_2}{qc} - \frac{2ay}{q} + 25\kappa\left(p^2g_2+p^{-1}g_{ab}+p^{-1}g_c\right) + O(n^{-1/2+4\delta}).$$

Similarly, we have

$$\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_i] \ge -\frac{5ad}{2qc} - \frac{6a^2z_2}{qc} - \frac{2ay}{q} - 25\kappa\left(p^2g_2+p^{-1}g_{ab}+p^{-1}g_c\right) + O(n^{-1/2+4\delta}). \quad (26)$$

We must also estimate the one-step change in $n^2(a+g_{ab})$, i.e. the deterministic part of $A^{\pm}_{u'u'',k'}$. We use Taylor's theorem with the Lagrange form of the remainder: for a function $h:\mathbb{R}\to\mathbb{R}$ twice differentiable on (x_0,x) and h' continuous on $[x_0,x]$, we have

$$h(x)-h(x_0) = h'(x_0)(x-x_0) + h''(x^*)(x-x_0)^2/2$$

for some $x^*\in[x_0,x]$. In our case, $x_0 = i/n^2 = t, x = (i+1)/n^2 = t+n^{-2}$. Thus for some $t^*\in[t,t+n^{-2}]$ we have

$$\Delta n^2(a+g_{ab}) = a'(t)+g'_{ab}(t) + \frac{a''(t^*)+g''_{ab}(t^*)}{2n^2} = a'(t)+g'_{ab}(t)+O(n^{-2}), \quad (27)$$

where the last expression follows from Propositions 1 and 2.

Putting (26) and (27) together we have

$$\begin{aligned}\mathbb{E}[\Delta A^+_{u'u'',k'}|\mathscr{F}_i] &\le -\frac{5ad}{2qc} - \frac{6a^2z_2}{qc} - \frac{2ay}{q} - a' - g'_{ab} + 25\kappa\left(p^2g_2 + p^{-1}g_{ab} + p^{-1}g_c\right) + O(n^{-1/2+4\delta})\\ &= -g'_{ab} + 25\kappa\left(p^2g_2 + p^{-1}g_{ab} + p^{-1}g_c\right) + O(n^{-1/2+4\delta})\\ &\le -5\kappa\left(p^2g_2 + p^{-1}g_{ab} + p^{-1}g_c\right) + O(n^{-1/2+4\delta})\\ &\le -\Omega\left(n^{-\omega}\right),\end{aligned}$$

where the second line follows from (10) which says $a' = -\frac{5ad}{2qc} - \frac{6a^2z_2}{qc} - \frac{2ay}{q}$, the third line follows from (17), and the final line follows from our choice of the error functions. Thus $A^+_{u'u'',k'}$ is a supermartingale. The reader can check that $A^-_{u'u'',k'}$ is a submartingale using an entirely "symmetric" calculation (i.e. we repeat the above calculation with the directions of inequalities reversed and the signs of the error terms reversed) using (26).

We will apply Freedman's inequality from Lemma 2. Our supermartingale will be $A^+_{u'u'',k'}$. First we determine a suitable value for D. Note that at each step i, the number of edges that have a color forbidden (when it was available at step $i-1$) is $O(n)$. Also, any edge has $O(1)$ colors forbidden at each step. Thus, the number of pairs (e,k) such that k was available at e at step $i-1$ but forbidden at step i is $O(n)$. But the only way for a pair (k,k') that is available at a triple (u,u',u'') at step $i-1$ to become forbidden at step i is to forbid one of the colors k,k' at one of the edges in $uu'u''$. Thus, we have $\Delta A_{u'u'',k'} = O(n)$.

Meanwhile we have by (27) and Propositions 1 and 2 that

$$\Delta n^2(a+g_{ab}) = a' + g'_{ab} + O(n^{-2}) = O(1)$$

and so

$$|\Delta A^+_{u'u'',k'}| \le |\Delta A_{u'u'',k'}| + |\Delta n^2(a+g_{ab})| = O(n).$$

Thus, using that $|\Delta A^+_{u'u'',k'}| = O(n)$ in the good event we get

$$\mathbf{Var}[\Delta A^+_{u'u'',k'}|\mathscr{F}_k] \le \mathbb{E}[(\Delta A^+_{u'u'',k'})^2|\mathscr{F}_k] = O(n)\cdot\mathbb{E}[|\Delta A^+_{u'u'',k'}||\mathscr{F}_k].$$

In order to bound $\mathbb{E}[|\Delta A^+_{u'u'',k'}||\mathscr{F}_k]$, first observe that

$$\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_k] = O(1) \qquad \text{and} \qquad -\mathbb{E}[\Delta A_{u'u'',k'}|\mathscr{F}_k] = O(1),$$

by (26), and hence

$$\mathbb{E}[|\Delta A^+_{u'u'',k'}||\mathscr{F}_k] \le \mathbb{E}[|\Delta A_{u'u'',k'}||\mathscr{F}_k] + \mathbb{E}[|\Delta n^2(a+g_{ab})||\mathscr{F}_k] = O(1).$$

Consequently, $\mathbf{Var}[\Delta A^+_{u'u'',k'}|\mathscr{F}_k] = O(n)$ and for all $i \le i_{max} < \frac{1}{6}n^2$ we have

$$V(i) = \sum_{0\le k\le i}\mathbf{Var}[A^+_{u'u'',k'}|\mathscr{F}_k] = O(n^3).$$

In view of the above calculations we are going to apply Freedman's inequality with $b = O(n^3)$ and $D = O(n)$.

We still need to estimate the initial value $A^+_{u'u'',k'}(0)$ of our supermartingale. Note that

$$A^+_{u'u'',k'}(0) = A_{u'u'',k'}(0) - n^2(a(0) + g_{ab}(0)). \tag{28}$$

Recall that $A_{u'u'',k'}(0)$ is the number of pairs (u,k) such that (k,k') is available at (u,u',u'') at step 0. The only requirement here is that $k' \in S_u$, and $k \notin S_u, S_{u'}, S_{u''}$. Observe that $A_{u'u'',k'}(0)$ is a binomial random variable with $A_{u'u'',k'}(0) \sim \mathrm{Bin}((n-2)|\textsc{col}|, s(1-s)^3)$. Thus, the expected value of $A_{u'u'',k'}(0)$ is

$$\mathbb{E}[A_{u'u'',k'}(0)] = (n-2)|\textsc{col}|s(1-s)^3 = n^2a(0) + O(n) \quad \text{and} \quad \mathbb{E}[A_{u'u'',k'}(0)] = \Theta(n^2)$$

so an easy application of the Chernoff bounds (2) and (3) tells us that a.a.s. $|A_{u'u'',k'}(0) - n^2a(0)| \le n^{3/2}$ for all u', u'', k'. Returning to (28) we have

$$A^+_{u'u'',k'}(0) \le n^{3/2} - n^2 g_{ab}(0) = n^{3/2} - n^{2-\omega} \le -\frac{1}{2}n^{2-\omega}.$$

Thus for our application of Freedman's inequality we get to use $\lambda = \frac{1}{2}n^{2-\omega}$. Freedman's inequality then gives us that the probability $A^+_{u'u'',k'}$ becomes positive before step i_{max} is at most

$$\exp\left(-\frac{\lambda^2}{2(b+D\lambda)}\right) = \exp\left\{-\Omega\left(\frac{\left(n^{2-\omega}\right)^2}{n^3 + n\cdot n^{2-\omega}}\right)\right\} = \exp\left\{-\Omega\left(n^{1-\omega}\right)\right\}.$$

Since there are $O(n^3)$ choices for u', u'', k', we have by the union bound that the probability any such choice ever sees $A^+_{u'u'',k'}$ become positive before step i_{max} is at most

$$O(n^3)\cdot \exp\left\{-\Omega\left(n^{1-\omega}\right)\right\} = o(1).$$

Similarly, one can apply Freedman's inequality to the supermartingales $-A^-_{u'u'',k'}$ to show that the probability any of them become positive before step i_{max} is $o(1)$. Thus, a.a.s. the good event $\mathcal{E}_{i_{max}}$ does not fail due to Condition (III).

Handling Condition (IV) is similar, since the type B variables are similar to type A (in particular they even have the same trajectory). To demonstrate the similarity, note that for $(u'', k') \in B_{uu',k}(i)$, the probability that $(u'', k') \notin B_{uu',k}(i+1)$ is

$$\mathbb{P}\left[\mathcal{L}_{u'',k} \cup \bigcup_{z=\{u',u''\}} \mathcal{L}_{z,k'} \quad \cup \mathcal{N}_{uu'',k} \cup \mathcal{N}_{u'u'',k'} \cup \bigcup_{e=\{uu'',u'u''\}} \mathcal{M}_{e,\bullet}\right].$$

And although the indices are different, there are exactly the same number of each of the events $\mathcal{L}_{z,k^*}$, $\mathcal{N}_{e,k^*}$, $\mathcal{M}_{e,\bullet}$, which will yield precisely the same estimates as $A_{u'u'',k'}$. Thus, to avoid too much repetition we will not show the work for Condition (IV).

8. Variable $C^{(1)}$

In this section, we address (V). Now define

$$C^{(1)\pm}_{uu'u''} = C^{(1)\pm}_{uu'u''}(i) := \begin{cases} C^{(1)}_{uu'u''} - n(c_1(t) \pm g_{c1}) & \text{if } \mathcal{E}_{i-1} \text{ holds,} \\ C^{(1)\pm}_{uu'u''}(i-1) & \text{otherwise.} \end{cases}$$

We demonstrate that $C^{(1)+}_{uu'u''}$ is a supermartingale. To estimate the one-step change, note that we may lose $k' \in C^{(1)}_{uu'u''}(i)$ if u', u'' is hit by k' or if $u'u''$ becomes part of an alternating $(u'u'', k')$-path. Thus, due to (25) and Claim 3, we get

$$\mathbb{E}[\Delta C^{(1)}_{uu'u''}|\mathcal{F}_i] = -\sum_{k' \in C^{(1)}_{uu'u''}} \mathbb{P}\left[\bigcup_{z\in\{u',u''\}} \left(\mathcal{L}_{z,k'} \cup \mathcal{N}_{u'u'',k'}\right)\right]$$

$$= - \sum_{k' \in C^{(1)}_{uu'u''}} \left[\sum_{z \in \{u', u''\}} \mathbb{P}(\mathcal{L}_{z,k'}) + \mathbb{P}(\mathcal{N}_{u'u'',k'}) + O(n^{-3+8\delta}) \right].$$

Now, Claim 1 yields

$$\begin{aligned}
\mathbb{E}[\Delta C^{(1)}_{uu'u''} | \mathcal{F}_i] \le & -n(c_1 - g_{c_1}) \Big\{ 2n^{-2} \left[\frac{5d}{6qc} - \kappa\left(p^{-8} g_{def} + p^{-6} g_c\right)\right] \\
& \qquad + n^{-2} \left[\frac{3az_2}{qc} - \kappa\left(p^{-3} g_2 + p^{-5} g_{ab} + p^{-5} g_c\right)\right]\Big\} + O(n^{-2+8\delta}) \\
\le & -n^{-1}(c_1 - g_{c_1}) \left[\frac{5d}{3qc} + \frac{3az_2}{qc} - 2\kappa(p^{-3} g_2 + p^{-5} g_{ab} + p^{-6} g_c + p^{-8} g_{def})\right] + O(n^{-2+8\delta}) \\
\le & \; n^{-1} \left(-\frac{5dc_1}{3qc} - \frac{3az_2 c_1}{qc} + 20\kappa(p^{-1} g_2 + p^{-3} g_{ab} + p^{-4} g_c + p^{-6} g_{def}) \right) + O(n^{-2+8\delta}),
\end{aligned}$$

where in the last line we use bounds from Claim 2 together with $g_1 = o(c_1)$ and $c_1 \le p^2$. The lower bound will follow by symmetric calculations.

Now by Taylor's theorem we have $\Delta(n(c_1 + g_{c_1})) = n^{-1}(c_1' + g_{c_1}') + O(n^{-3})$. Therefore, in the good event by applying (11) and (18), we obtain

$$\begin{aligned}
\mathbb{E}[\Delta C^{(1)+}_{uu'u''} | \mathcal{F}_i] \le & \; n^{-1} \Big(-c_1' - \frac{5dc_1}{3qc} - \frac{3az_2 c_1}{qc} - g_{c_1}' \\
& \qquad + 20\kappa(p^{-1} g_2 + p^{-3} g_{ab} + p^{-4} g_c + p^{-6} g_{def}) \Big) + O(n^{-2+8\delta}) \\
= & \; n^{-1} \left(-g_{c_1}' + 20\kappa(p^{-1} g_2 + p^{-3} g_{ab} + p^{-4} g_c + p^{-6} g_{def}) \right) + O(n^{-2+8\delta}) \\
\le & \; n^{-1} \left(-10\kappa(p^{-1} g_2 + p^{-3} g_{ab} + p^{-4} g_c + p^{-6} g_{def}) \right) + O(n^{-2+8\delta}) \\
\le & -\Omega(n^{-1-\omega}).
\end{aligned}$$

Now to apply Freedman's inequality, we estimate the maximum one-step change of $C^{(1)}_{uu'u''}$. Since the number of ways to forbid a color at an edge in one step is $O(1)$, we get that that $|\Delta C^{(1)}_{uu'u''}| = \Delta C^{(1)}_{uu'u''} = O(1)$, and by Propositions 1 and 2, $\Delta n(c_1 + g_{c_1}) = n^{-1}(c_1' + g_{c_1}') + O(n^{-3})$ yielding

$$|\Delta n(c_1 + g_{c_1})| \le n^{-1}(|c_1'| + |g_{c_1}'|) + O(n^{-3}) = O(n^{-1}|c_1'|) = O(n^{-1})$$

and so

$$|\Delta C^{(1)+}_{uu'u''}| \le |\Delta C^{(1)}_{uu'u''}| + |\Delta n(c_1 + g_{c_1})| = O(1).$$

Thus we let $D = O(1)$ in Freedman's inequality. Further, we have

$$\mathbf{Var}[\Delta C^{(1)+}_{uu'u''} | \mathcal{F}_k] \le \mathbb{E}[(\Delta C^{(1)+}_{uu'u''})^2 | \mathcal{F}_k] = O(1) \cdot \mathbb{E}[|\Delta C^{(1)+}_{uu'u''}| | \mathcal{F}_k] = O(n^{-1}).$$

Therefore, $V(i) = O(n)$ for all $i \le i_{max}$ and so we take $b = O(n)$. In addition, Chernoff's bound allows us to take $\lambda = \frac{1}{2} n^{1-\omega}$, and so Freedman's inequality demonstrates that the probability that $C^{(1)+}_{uu'u''}$ becomes positive before step i_{max} is at most $\exp\left\{-\Omega\left(n^{1-2\omega}\right)\right\}$, which beats the union bound over all $O(n^3)$ choices for u, u' and u''.

9. VARIABLE D

In this section, we address (VII). Since (VIII)–(IX) are very similar and these variables share the same trajectory, we will omit their calculations (see our discussion of B type variables in Section 7). Define

$$D^{\pm}_{u,k}=D^{\pm}_{u,k}(i):=\begin{cases}D_{u,k}-n^3(d(t)\pm g_{def}) & \text{if } \mathcal{E}_{i-1} \text{ holds,}\\ D^{\pm}_{u,k}(i-1) & \text{otherwise.}\end{cases}$$

To bound the expected one-step change, note that we can lose $(u',u'',k')\in D_{u,k}$ in several ways: one of the edges could become matched, u' or u'' could become hit by k or k', or one of the edges could become part of an alternating path. Hence, using Claims 3, 1 and 2 together with $g_{def}=o(d)$, and $d\le p^7$, yield

$$\begin{aligned}
\mathbb{E}[\Delta D_{u,k}|\mathcal{F}_i] &= -\sum_{(u',u'',k')\in D_{u,k}}\mathbb{P}\left(\bigcup_{e\in\{uu',uu'',u'u''\}}\mathcal{M}_{e,\bullet}\cup\bigcup_{z\in\{u',u''\}}((\mathcal{L}_{z,k})\cup(\mathcal{L}_{z,k'}))\cup\bigcup_{e\in\{uu',uu''\}}\mathcal{N}_{e,k}\cup\mathcal{N}_{u'u'',k'}\right)\\
&= -\sum_{(u',u'',k')\in D_{u,k}}\left(\sum_{e\in\{uu',uu'',u'u''\}}\mathbb{P}(\mathcal{M}_{e,\bullet})+\sum_{z\in\{u',u''\}}(\mathbb{P}(\mathcal{L}_{z,k})+\mathbb{P}(\mathcal{L}_{z,k'}))\right.\\
&\qquad\left.+\sum_{e\in\{uu',uu''\}}\mathbb{P}(\mathcal{N}_{e,k})+\mathbb{P}(\mathcal{N}_{u'u'',k'})+O(n^{-3+12\delta})\right)\\
&\le -n^3(d-g_{def})\left\{4n^{-2}\left[\frac{5d}{6qc}-\kappa\left(p^{-8}g_{def}+p^{-6}g_c\right)\right]\right.\\
&\qquad\left.+\ 3n^{-2}\left[\frac{3az_2}{qc}+\frac{y}{q}-\kappa\left(p^{-3}g_2+p^{-5}g_{ab}+p^{-5}g_c\right)\right]+O(n^{-5/2+4\delta})\right\}+O(n^{12\delta})\\
&\le -(d-g_{def})\left[\frac{20d}{6qc}+\frac{9az_2}{qc}+\frac{3y}{q}-7\kappa\left(p^{-3}g_2+p^{-8}g_{def}+p^{-5}g_{ab}+p^{-6}g_c\right)\right]n+O(n^{1/2+4\delta})\\
&\le\left[-\frac{20d^2}{6qc}-\frac{9az_2d}{qc}-\frac{3yd}{q}+20\kappa\left(p^4g_2+p^{-1}g_{def}+p^2g_{ab}+pg_c\right)\right]n+O(n^{1/2+4\delta}).
\end{aligned}$$

On the other hand, by Taylor's theorem we have $\Delta(n^3(d+g_{def}))=(d'+g'_{def})n+O(n^{-1})$. Therefore, in the good event due to (13) and (20), we get

$$\begin{aligned}
\mathbb{E}[\Delta D^+_{u,k}|\mathcal{F}_i] &\le\left[-d'-\frac{20d^2}{6qc}-\frac{9az_2d}{qc}-\frac{3yd}{q}-g'_{def}+20\kappa\left(p^4g_2+p^{-1}g_{def}+p^2g_{ab}+pg_c\right)\right]n+O(n^{1/2+4\delta})\\
&=\left[-g'_{def}+20\kappa\left(p^4g_2+p^{-1}g_{def}+p^2g_{ab}+pg_c\right)\right]n+O(n^{1/2+4\delta})\\
&\le\left[-10\kappa\left(p^4g_2+p^{-1}g_{def}+p^2g_{ab}+pg_c\right)\right]n+O(n^{1/2+4\delta})\\
&\le-\Omega(n^{1-\omega}).
\end{aligned}$$

As before, we apply Freedman's inequality to $D^+_{u,k}$ by first estimating the maximum one-step change of $D_{u,k}$. As discussed above, the maximum one-step change is $O(n^2)$ by having at most $O(n)$ edges e forbid the $O(n)$ pairs (e,k'). In addition, Propositions 1 and 2 imply

$$|\Delta n^3(d+g_{def})|\le n(|d'|+|g'_{def}|)+O(n^{-1})=O(n|d'|)=O(n)$$

and so

$$|\Delta D_{u,k}^{+}| \le |\Delta D_{u,k}| + |\Delta n^3(d + g_{def})| = O(n^2),$$

so we take $D = O(n^2)$. Furthermore,

$$\mathbf{Var}[\Delta D_{u,k}^{+}|\mathscr{F}_k] \le \mathbb{E}[(\Delta D_{u,k}^{+})^2|\mathscr{F}_k] = O(n^2)\cdot \mathbb{E}[|\Delta D_{u,k}^{+}||\mathscr{F}_k] = O(n^3)$$

and so for all $i \le i_{max} < \frac{1}{6}n^2$ we have $V(i) = O(n^5)$.

Therefore, take $b = O(n^5)$ and $\lambda = \frac{1}{2}n^{3-\omega}$ to get by Freedman's inequality and the union bound a failure probability of $O(n^2)\cdot \exp(-\Omega(n^{1-2\omega})) = o(1)$.

10. Variable Z_0

In this section, we address (X) by considering Z_0. Extending the work on the remaining variables Z_1 and Z_2 requires some similar calculations (some of which involve the bounds in (XIII)), which we omit for readability.

Define

$$Z^{\pm}_{uv,k,0,0,0} = Z^{\pm}_{uv,k,0,0,0}(i) := \begin{cases} Z_{uv,k,0,0,0} - n^3(z_0(t) \pm g_0) & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ Z^{\pm}_{uv,k,0,0,0}(i-1) & \text{otherwise.} \end{cases}$$

Notice that the expected one-step change can never increase and we may lose a $(x,y,k') \in Z_{uv,k,0,0,0}$ in several ways: the vertex x,y is hit with the color k or x,y,u,v is hit with the color k'; or the edge xy, ux, vy is colored; or ux, vy, xy becomes part of an alternating path. Thus, by Claims 3, 1 and 2 together with bounds $g_0 = o(z_0)$, and $z_0 \le p^9$, we get

$$\begin{aligned}
\mathbb{E}[\Delta Z_{uv,k,0,0,0}|\mathscr{F}_i] &= -\sum_{(x,y,k')\in Z_{uv,k,0,0,0}} \mathbb{P}\Bigg(\bigcup_{z\in\{x,y\}} \mathscr{L}_{z,k} \cup \bigcup_{z\in\{x,y,u,v\}} \mathscr{L}_{z,k'} \\
&\qquad \cup \bigcup_{e\in\{xy,ux,vy\}} \mathscr{M}_{e,\bullet} \cup \bigcup_{e\in\{ux,vy\}} \mathscr{N}_{e,k'} \cup \mathscr{N}_{xy,k}\Bigg) \\
&= -\sum_{(x,y,k')\in Z_{uv,k,0,0,0}} \Bigg(\sum_{z\in\{x,y\}} \mathbb{P}(\mathscr{L}_{z,k}) + \sum_{z\in\{x,y,u,v\}} \mathbb{P}(\mathscr{L}_{z,k'}) \\
&\qquad + \sum_{e\in\{xy,ux,vy\}} \mathbb{P}(\mathscr{M}_{e,\bullet}) + \sum_{e\in\{ux,vy\}} \mathbb{P}(\mathscr{N}_{e,k'}) + \mathbb{P}(\mathscr{N}_{xy,k}) + O(n^{-3+12\delta})\Bigg) \\
&\le -n^3(z_0 - g_0)\Bigg\{6n^{-2}\left[\frac{5d}{6qc} - \kappa\left(p^{-8}g_{def} + p^{-6}g_c\right)\right] \\
&\qquad +3n^{-2}\left[\frac{3az_2}{qc} + \frac{y}{q} - \kappa\left(p^{-3}g_2 + p^{-5}g_{ab} + p^{-5}g_c\right) + O(n^{-5/2+4\delta})\right]\Bigg\} + O\left(n^{12\delta}\right) \\
&= -n(z_0 - g_0)\Bigg[\frac{5d}{qc} + \frac{9az_2}{qc} + \frac{3y}{q} \\
&\qquad -9\kappa\left(p^{-3}g_2 + p^{-8}g_{def} + p^{-5}g_{ab} + p^{-6}g_c\right)\Bigg] + O\left(n^{1/2+4\delta}\right) \\
&\le n\left[-\frac{5dz_0}{qc} - \frac{9az_2z_0}{qc} - \frac{3yz_0}{q} + 20\kappa\left(p^6g_2 + pg_{def} + p^4g_{ab} + p^3g_c\right)\right] + O\left(n^{1/2+4\delta}\right).
\end{aligned}$$

By Taylor's theorem we have $\Delta(n^3(z_0+g_0))=(z_0'+g_0')n+O(n^{-1})$. Therefore in the good event by (14) and (21), we obtain

$$\begin{aligned}\mathbb{E}[\Delta Z^+_{uv,k,0,0,0}|\mathscr{F}_i] &\le \Bigg[-z_0'-\frac{5dz_0}{qc}-\frac{9az_2z_0}{qc}-\frac{3yz_0}{q}-g_0' \\ &\qquad +20\kappa\left(p^6g_2+pg_{def}+p^4g_{ab}+p^3g_c\right)\Bigg]n+O\left(n^{1/2+4\delta}\right) \\ &=\left[-g_0'+20\kappa\left(p^6g_2+pg_{def}+p^4g_{ab}+p^3g_c\right)\right]n+O\left(n^{1/2+4\delta}\right) \\ &\le\left[-10\kappa\left(p^6g_2+pg_{def}+p^4g_{ab}+p^3g_c\right)\right]n+O\left(n^{1/2+4\delta}\right) \\ &\le -\Omega(n^{1-\omega}).\end{aligned}$$

Consider $Z_{uv,k,0,0,0}$ for some fixed edge uv and color k. The one-step change in this random variable never has any positive contributions, and its negative contributions can come in several ways. Suppose $(x,y,k')\in Z_{uv,k,0,0,0}(i)$. Then we could have $(x,y,k')\notin Z_{uv,k,0,0,0}(i+1)$ for any of the following (exhaustive) list of reasons:

(i) one of the edges ux, xy, yv gets colored,
(ii) one of the vertices x, y gets hit by one of the colors k, k',
(iii) one of the vertices u, v gets hit by k',
(iv) k is forbidden at xy through an alternating 4-cycle,
(v) k' is forbidden at ux or yv through an alternating 4-cycle.

Consider the triples (x,y,k') that are removed from $Z_{uv,k,0,0,0}$ due to (i). Two of the vertices in $\{u,x,y,v\}$ must be in the triangle that gets colored in this step, and so the number of triples (x,y,k') is at most $O(n^2)$. Reason (ii) is similarly $O(n^2)$. Reason (iii) is $O(n^2)$ since k' must be one of the colors in the triangle getting colored. Now in (iv), we observe that for a fixed color k, in a single step k is forbidden at $O(n)$ many edges due to potential 4-cycles. Since xy would have to be one of those edges, we get $O(n^2)$. Now for (v), observe that for a fixed color k', there are at most $O(1)$ edges ux adjacent to u such that k' is forbidden at ux through a potential 4-cycle. Thus, we obtain again $O(n^2)$.

We now apply Freedman's inequality. Note that

$$|\Delta n^3(z_0+g_0)|\le n(|z_0'|+|g_0'|)+O(n^{-1})=O(n|z_0'|)=O(n^{1+8\delta})$$

and so

$$|\Delta Z^+_{uv,k,0,0,0}|\le|\Delta Z_{uv,k,0,0,0}|+|\Delta n^3(z_0+g_0)|=O(n^2).$$

Therefore, we let $D=O(n^2)$. In addition,

$$\mathbf{Var}[\Delta Z^+_{uv,k,0,0,0}|\mathscr{F}_k]\le\mathbb{E}[(\Delta Z^+_{uv,k,0,0,0})^2|\mathscr{F}_k]\le O(n^2)\cdot\mathbb{E}[|\Delta Z^+_{uv,k,0,0,0}||\mathscr{F}_k]=O(n^3)$$

implying that $V(i)=O(n^5)$. Therefore, take $b=O(n^5)$. Using Chernoff's bound to estimate $Z^+_{uv,k,0,0,0}(0)$ allows us to set $\lambda=\frac{1}{2}n^{3-\omega}$. Thus Freedman's inequality gives us an exponentially small failure probability.

11. Bounds on Ξ, Φ, Ψ, Λ

In this section we bound the probability that the good event $\mathscr{E}_{i_{max}}$ fails due to Condition (XIII). The variables we are bounding here are all similar, so we will only show the details for

$\Xi_{u,v,k}$. First we define versions of these variables that are "frozen" outside the good event $\mathscr{E}_{i-1}$:

$$\Xi^*_{u,v,k}(i) := \begin{cases} \Xi_{u,v,k}(i) & \text{if } \mathscr{E}_{i-1} \text{ holds,} \\ \Xi^*_{u,v,k}(i-1) & \text{otherwise.} \end{cases}$$

Note that we have $\Delta\Xi_{u,v,k}(i) = O(1)$ and therefore $\Delta\Xi^*_{u,v,k}(i) = O(1)$. We bound the probability that $\Delta\Xi_{u,v,k}(i) \neq 0$ as follows. First we bound the number of "predecessors," (see figures below) i.e. triples (x, y, k') which are not in $\Xi_{u,v,k}(i-1)$ but which could become an element of $\Xi_{u,v,k}(i)$. On the first row in Figure 10 below we have "single-edge predecessors" that only need one edge colored in order to become part of $\Xi_{u,v,k}(i)$. On the second row we see "double-edge predecessors" which need two edges colored simultaneously (of course, for two edges to get colored in one step they would need to be adjacent).

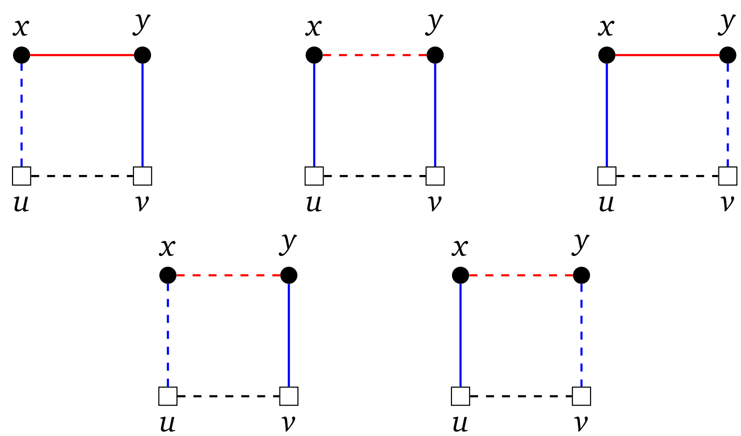


FIGURE 10. Depictions of "double-edge predecessors" (on the first row) and "single-edge predecessors" (on the second row) of $\Xi_{u,v,k}(i)$.

Note that the number of single-edge predecessors is $O(n)$ since for each fixed k' there is a constant number of choices for x, y. For a single-edge predecessor triple (x, y, k') to become part of $\Xi_{u,v,k}(i)$, a particular edge needs to get a particular color, which has probability $O\left(n^{-3+3\delta}\right)$ in the good event. The number of double-edge predecessors is $O(n^2)$, and for one of them to become part of $\Xi_{u,v,k}(i)$ we need to color a particular triangle using a particular pair of colors, which has probability $O\left(n^{-5+8\delta}\right)$ in the good event. Thus, in the good event we have

$$\mathbb{P}\left(\Delta\Xi_{u,v,k}(i) \neq 0\right) = O(n) \cdot O\left(n^{-3+3\delta}\right) + O(n^2) \cdot O\left(n^{-5+8\delta}\right) = O\left(n^{-2+3\delta}\right).$$

Of course this implies $\mathbb{P}\left(\Delta\Xi^*_{u,v,k}(i) \neq 0\right) = O\left(n^{-2+3\delta}\right)$ as well. Thus, the final value $\Xi^*_{u,v,k}(i_{max})$ is stochastically dominated by $X \sim K\text{Bin}(i_{max}, Kn^{-2+3\delta})$ for some constant K. An easy application of Chernoff shows that

$$\mathbb{P}\left(X > n^{4\delta}\right) \leq \exp\left\{-\Omega(n^{4\delta})\right\}.$$

Since there are only a polynomial number of variables $\Xi_{u,v,k}$, the union bound shows that a.a.s. none of them exceed $n^{4\delta}$.

12. Finishing the coloring

In this section we describe Phase 2 of our coloring procedure. We assume that Phase 1 has terminated successfully (i.e. the event $\mathcal{E}_{i_{max}}$ holds). In Phase 2 we will assign to each uncolored edge a uniform random color from the $\varepsilon n/2$ colors in $\overline{\text{COL}} \setminus \text{COL}$. We will use the Lovász Local Lemma (Lemma 3) to show there is a positive probability that none of the following "bad" events occur in Phase 2. Here when we say "uncolored edges" we mean edges that were not colored in Phase 1. Define the following events:

- For two adjacent uncolored edges e_1, e_2 let $B_1(e_1, e_2)$ be the event that both edges get the same color.
- For any 4-cycle of uncolored edges e_1, e_2, e_3, e_4, let $B_2(e_1, e_2, e_3, e_4)$ be the event that this 4-cycle becomes alternating (i.e. e_1 gets the same color as e_3, and e_2 gets the same color as e_4).
- For any 4-cycle of edges e_1, e_2, e_3, e_4 such that e_1 and e_3 are uncolored and e_2 and e_4 were given the same color in Phase 1, let $B_3(e_1, e_3)$ be the event that this 4-cycle becomes alternating (i.e. e_1 gets the same color as e_3).

Let $\mathcal{B}$ be the family of all bad events of types B_1, B_2, B_3 described above. Note that if none of the events in $\mathcal{B}$ happens, then Phase 2 gives us a $(4,5)$-coloring.

Toward describing our dependency graph we claim the following:

Claim 4. *Fix any event $B \in \mathcal{B}$ (of any type B_1, B_2 or B_3). Among the other events in $\mathcal{B}$, B is mutually independent with all but at most*

- *$O(n^{1-\delta})$ events of type B_1,*
- *$O(n^{2-2\delta})$ events of type B_2, and*
- *$O(n^{1-\delta})$ events of type B_3.*

Proof. Every event in $\mathcal{B}$ involves some set of uncolored edges and the colors they get in Phase 2. Any such event B is mutually independent of the set of all events B' that do not involve any of the same edges as B. So, for each type (i.e. type B_1, B_2, or B_3) we bound the number of B' of that type sharing an edge with B.

We show that any fixed uncolored edge e_1 is in $O(n^{1-\delta})$ events of the form $B_1(e_1, e_2)$. Indeed, this will follow from bounding the number of uncolored edges at a vertex. Bohman, Frieze and Lubetzky [6] proved that in the triangle removal process the degree of each vertex is a.a.s. $(1+o(1))np$ as long as we have, say, $p \ge n^{-1/3}$ (the power of n could be any constant larger than $-1/2$). In our analysis we are requiring the stronger condition $p \ge n^{-\delta}$ (which is the value of p at step i_{max} when we stopped the Phase 1 process). Thus, at the end of Phase 1 each vertex is incident with $O(n^{1-\delta})$ uncolored edges.

Next we show that any fixed uncolored edge e_1 is in $O(n^{2-\delta})$ events of the form $B_2(e_1, e_2, e_3, e_4)$. But, given e_1 and our bound on degrees, there are $O(n^{1-\delta})$ choices for e_2 and then $O(n^{1-\delta})$ choices for e_3, which determines at most one choice for e_4 and we are done.

Finally, we show that any fixed uncolored edge e_1 is in $O(n^{1-\delta})$ events of the form $B_3(e_1, e_3)$. We know that at the end of Phase 1 we have

$$\sum_{k \in \text{COL}} Z_{e_1,k,1,0,1} = O\left(|\text{COL}| \cdot n^{1-3\delta}\right) = O\left(n^{2-3\delta}\right).$$

Each event $B_3(e_1, e_3)$ is counted in the above sum once for every color k available at e_3. So we estimate the number of colors available at an edge. Say u', u'' are the endpoints of e_3. We know that

$$\sum_{u\in V} C^{(1)}_{uu'u''} = \Theta\left(n \cdot n^{1-2\delta}\right) = \Theta\left(n^{2-2\delta}\right).$$

For each color k' available at e_3, the sum above counts k' once for every vertex u such that $k' \in S_u$. An easy application of the Chernoff bound gives us that a.a.s. for every color k' there are $(1+o(1))ns = \Theta(n)$ vertices u such that $k' \in S_u$. Thus the number of colors available at e_3 is $\Theta\left(n^{1-2\delta}\right)$. Thus, the number edges e_3 such that we have a bad event $B(e_1, e_3)$ is

$$O\left(\frac{n^{2-3\delta}}{n^{1-2\delta}}\right) = O\left(n^{1-\delta}\right),$$

as required. □

To apply the Local Lemma we must assign to each bad event $B \in \mathscr{B}$ a number $x_B \in [0,1)$. To all the events of type B_j we assign the number x_j $(j = 1,2,3)$, where

$$x_1 := \frac{10}{\varepsilon n}, \qquad x_2 := \frac{10}{(\varepsilon n)^2}, \qquad x_3 := \frac{10}{\varepsilon n}.$$

We check the condition (4) of the Local Lemma. Since Phase 2 uses the set $\overline{\text{COL}} \setminus \text{COL}$ of $\varepsilon n/2$ colors, the probability of any B_1 event is $2/(\varepsilon n)$, which is smaller than

$$x_1(1-x_1)^{O(n^{1-\delta})}(1-x_2)^{O(n^{2-2\delta})}(1-x_3)^{O(n^{1-\delta})} = (1+o(1))x_1.$$

The probability of any B_2 event is $4/(\varepsilon n)^2$, which is smaller than

$$x_2(1-x_1)^{O(n^{1-\delta})}(1-x_2)^{O(n^{2-2\delta})}(1-x_3)^{O(n^{1-\delta})} = (1+o(1))x_2.$$

The probability of any B_3 event is $2/(\varepsilon n)$, which is smaller than

$$x_3(1-x_1)^{O(n^{1-\delta})}(1-x_2)^{O(n^{2-2\delta})}(1-x_3)^{O(n^{1-\delta})} = (1+o(1))x_3.$$

Thus, the conditions of Lemma 3 are met and so with positive probability Phase 2 gives us a $(4,5)$-coloring. This completes the proof of Theorem 1.

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