Localization Game for Random Geometric Graphs

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Abstract

The localization game is a two player combinatorial game played on a graph $G = (V, E)$. The cops choose a set of vertices $S_1 \subseteq V$ with $|S_1| = k$. The robber then chooses a vertex $v \in V$ whose location is hidden from the cops, but the cops learn the graph distance between the current position of the robber and the vertices in S_1. If this information is sufficient to locate the robber, the cops win immediately; otherwise the cops choose another set of vertices $S_2 \subseteq V$ with $|S_2| = k$, and the robber may move to a neighbouring vertex. The new distances to the robber are presented, and if the cops can deduce the new location of the robber based on all information they accumulated thus far, then they win; otherwise, a new round begins. If the robber has a strategy to avoid being captured, then she wins. The localization number is defined to be the smallest integer k so that the cops win the game. In this paper we determine the localization number (up to poly-logarithmic factors) of the random geometric graph $G \in \mathcal{G}(n, r)$ slightly above the connectivity threshold.

Keywords: random graphs, random geometric graphs, localization number, cops and robbers
MSC Class: 05C80, 60C05, 05D40, 60D05

1 Introduction

1.1 Localization game

Graph searching focuses on the analysis of games and graph processes that model some form of intrusion in a network and efforts to eliminate or contain that intrusion. One of the best known examples of graph searching is the game of *Cops and Robbers*, wherein a robber is loose on the network and a set of cops attempts to capture the robber. For a book on graph searching see [4].

In this paper we consider the *Localization Game* that is related to the well studied *Cops and Robbers* game. For a fixed integer $k \geq 1$, the localization game with k sensors is a two player combinatorial game played on a graph G which is known to both players. To initialize the game, the *cops* first choose a set $S_1 \subseteq V(G)$ with $|S_1| = k$. The *robber* then chooses a vertex $v \in V(G)$ to start at, whose location on the graph is hidden from the cops. The cops then learn the graph distance between the current position of the robber and the vertices of S_1. If this information is sufficient to locate the robber, then the cops win immediately. Otherwise, a new round begins, and the cops now choose another subset $S_2 \subseteq V(G)$ of size k, based on all the past information available to them. At this point, the robber is allowed to

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move to any vertex of distance one from its current location, based on S_1 *and* S_2. The distances of the robber's new location to the vertices of S_2 are then presented to the cops, at which point the cops win if these new distance values in conjunction with the previous ones are sufficient to locate the robber. If the cops' information is still insufficient to win the game, then another round begins. These rounds continue until the cops are able to locate the robber, in which case we say that the cops win, or the game proceeds indefinitely, in which case we say that the robber wins. Hence, to summarize, each round consists of the following *steps*:

a) the cops place k sensors on some vertices of G,

b) the robber moves to a neighbor of the vertex she currently occupies or stays put (if this is the first round, then she simply chooses any vertex of G to start with),

c) the cops obtain the information about the distances between the sensors and the robber,

d) the cops combine the information from all rounds so far and the game ends if this is enough to detect the position of the robber.

We provide more details in Subsection 2.1 to show that the localization game is a combinatorial game. This motivates the following definition. Given a graph G, its localization number, denoted by $\zeta(G)$, is the minimum k such that the cops can eventually locate the robber using exactly k sensors in each round. The localization game was introduced for one sensor ($k = 1$) in [18, 19] and was further studied in [3, 5, 6, 7, 12].

Let us emphasize that the cops only win provided their strategy beats all robber's strategies, and thus is a worst-case win condition. An alternative "robber first" definition of the localization game involves the robber moving first in each round, in particular choosing their move prior to the initial placement of the cops' sensors. Since both games require a worst case guarantee for the cops to win, these games are equivalent.

1.2 Random geometric graphs

In this paper we investigate geometric graphs in the plane. Given a positive integer n and a threshold distance $r > 0$, we consider the *random geometric graph* $G \in \mathcal{G}(n, r)$ on vertex set $V = \{v_1, v_2, \dots, v_n\}$ obtained by starting with n random points $x_1, x_2, \dots, x_n$ in $\mathbb{R}^2$ sampled independently and uniformly in the square $[0, \sqrt{n}]^2$. For any $i \neq j$, the vertices v_i and v_j are adjacent when the Euclidean distance $d_E(x_i, x_j)$ is at most r. Note that, with probability 1, no point in $[0, \sqrt{n}]^2$ is chosen more than once, so we may identify each vertex $v_i \in V$ with its corresponding geometric position x_i. In fact, in order to simplify some of the proofs, we will work with the random geometric graph $G \in \mathcal{T}(n, r)$ equipped with the torus metric $d_T(\cdot, \cdot)$ instead of $d_E(\cdot, \cdot)$. For more details about these models see, for example, [17].

Our results are asymptotic in nature. In other words, we will assume that $n \to \infty$ and $r = r(n)$ may (and usually does) tend to infinity as $n \to \infty$. We are interested in events that hold *asymptotically almost surely* (*a.a.s.*), that is, events that hold with probability tending to 1 as $n \to \infty$. It is known that $r_c = r_c(n) = \sqrt{\frac{\log n}{\pi}}$ is a sharp threshold function for connectivity for $G \in \mathcal{G}(n, r)$ (see, for example, [11, 16]). This means that for every $\varepsilon > 0$, if $r \leq (1 - \varepsilon) r_c$, then G is disconnected a.a.s., whilst if $r \geq (1 + \varepsilon) r_c$, then G is connected a.a.s. The same property holds for $G \in \mathcal{T}(n, r)$.

1.3 Asymptotic notation

Given two functions $f = f(n)$ and $g = g(n)$, we will write:

- $f(n) = O(g(n))$ if there exists an absolute constant $c \in \mathbb{R}_+$ such that $|f(n)| \leq c|g(n)|$ for all n,
- $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$,

- $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$,
- $f(n) = o(g(n))$ or $f(n) \ll g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and
- $f(n) \gg g(n)$ if $g(n) = o(f(n))$.

1.4 Our main result

Here and below, we fix $r_0 = r_0(n) = 70\sqrt{\log n}$. We split the statement of Theorem 1.1 into four different cases corresponding to different proof strategies (or adaptations of proof strategies) for different values of the parameter r. Unfortunately, our proofs do not give insight which (if any) of the obtained bounds are tight.

Theorem 1.1. *Fix $r = r(n) \in [r_0, \sqrt{n}/4)$ and let $G \in \mathcal{T}(n,r)$. Then, a.a.s. the following bounds hold:*

1. *If $\log^{3/2} n \le r < \sqrt{n}/4$, then $\Omega(r^{4/3}/(\log n)^{1/3}) = \zeta(G) = O(r^{4/3})$.*
2. *If $\log n \le r < \log^{3/2} n$, then $\Omega(r^{4/3}/(\log n)^{1/3}) = \zeta(G) = O(\log^2 n)$.*
3. *If $\frac{\log n}{(\log\log n)^{1/2}\log\log\log n} \le r < \log n$, then $\Omega(r^2/\log n) = \zeta(G) = O(r^2)$.*
4. *If $r_0 \le r < \frac{\log n}{(\log\log n)^{1/2}\log\log\log n}$, then $\Omega(\log n/\log(r^2/\log n)) = \zeta(G) = O(r^2)$.*

Note that the lower bound we prove below for $\frac{\log n}{(\log\log n)^{1/2}\log\log\log n} \le r < \log n$ in Theorem 1.1 is slightly stronger, namely $\zeta(G) = \Omega(r^2\log(e\log n/r)/\log n)$, but for the sake of readability we opted here for a slightly weaker version. Next, let us point out that we restrict ourselves to $r < \sqrt{n}/4$. This is done for a technical reason. For extremely dense graphs, the behaviour of $\mathcal{T}(n,r)$ changes drastically. In the extreme case, when $r \ge \sqrt{n/2}$, $\mathcal{T}(n,r)$ is simply the complete graph on n vertices and $\zeta(\mathcal{T}(n,r)) = n-1$. Such dense graphs are not so interesting as they do not represent the typical nature of random geometric graphs and for that reason are rarely studied. Indeed, for such dense graphs the effect of wrapping around the torus has to be considered, and the results for $\mathcal{T}(n,r)$ typically differ from the ones for $\mathcal{G}(n,r)$, the random geometric graph in the square.

1.5 Main ideas behind the proofs

The proof of Theorem 1.1 in divided into a proof of the upper bounds and a proof of the lower bounds in the four regimes. For the upper bounds, we first show that by using only four sensors the cops may locate the position of the robber throughout several rounds within a square S of side length $20000r$. Then, the cops need one last round to win. Roughly speaking, they divide their set of sensors into two parts of comparable sizes. Then, they distribute the first part of their sensors uniformly at random among all vertices of G in the square S. Finally, the cops take care of the vertices in the square, which cannot be uniquely distinguished by the sensors already used, and put one sensor on any such vertex.

For the lower bounds, we show that it is sufficient for the robber to choose any ball of radius $r/3$ before even knowing the random graph $G \in \mathcal{T}(n,r)$. Once having done that, we prove that a.a.s. the number of sensors, given by Theorem 1.1, is not sufficient to distinguish the position of the robber even if she decides to stay in the ball forever.

1.6 Related results

The metric dimension of a graph G, written $\beta(G)$, is the minimum number of sensors needed in the localization game so that the cops can win in one round. The localization number is related to the metric dimension of a graph in a way that is analogous to how the cop number is related to the domination number. In particular, it follows that $\zeta(G) \le \beta(G)$, but in many cases this inequality is far from tight.

Although the game has not yet been studied for random geometric graphs, there are some known results for the classical *binomial random graph* $\mathcal{G}(n,p)$. The localization number for dense random graphs (in particular, in the regime in which $\mathcal{G}(n,p)$ has diameter two a.a.s.) was studied in [10]. The bounds for dense graphs were consecutively improved in [9], and the arguments were extended to sparse graphs.

The metric dimension was also studied for the $\mathcal{G}(n,p)$ model. The statements of the bounds for $\beta(G)$ with $G \in \mathcal{G}(n,p)$ obtained in [2] are slightly technical, but the following observations can be made: for sparser graphs (that is, graphs of diameter at least three a.a.s., which corresponds to $i \geq 2$ in the discussion below), it follows from [2] and [9] that $\zeta(G) < \beta(G)$. In fact, if $np = n^{x+o(1)}$ for some $x \in (\frac{1}{i+1}, \frac{1}{i})$, $i \in \mathbb{N} \setminus \{1\}$, then a.a.s. $i + o(1) \leq \beta(G)/\zeta(G) \leq 1/x + o(1) < i + 1$, and so these two graph parameters are a multiplicative constant away from each other (the ratio being roughly equal to the diameter of the graph). Moreover, for very sparse graphs, say for example $np = \log^6 n$, a.a.s. $\zeta(G) = \Theta(n \log\log n/(np)^i)$ whereas $\beta(G) = \Theta(n \log n/(np)^i)$, implying that for such value of np, $\zeta(G) = o(\beta(G))$.

2 Preliminaries

2.1 Reformulation of the game with perfect information for the cops

In this section we show that the game we study is a combinatorial, perfect information game despite the fact that the robber is invisible for the cops. Let $G = (V, E)$ be a connected graph. Given a set $S \subseteq V$ of size k, $S = \{s_1, s_2, \ldots, s_k\}$, and a vertex $v \in V$, the *S-signature* of v is defined as the vector $\mathbf{d} = \mathbf{d}(S, v) = (d_1, d_2, \ldots, d_k)$ where for every $i \in \{1, 2, \ldots, k\}$, $d_i = d_G(s_i, v)$ is the graph distance from s_i to v. Given a set $V' \subseteq V$, let

$$N[V'] = N_G[V'] := \{v \in V : d(v, u) \leq 1 \text{ for some } u \in R\},$$

that is, $N[V']$ is the closed neighborhood of the set of vertices V' in G.

The *localization game with k sensors* is a game played by two players, the *cops* and the *invisible robber*. While playing the game, both the cops and the robber are aware of the underlying graph and each of the previous moves of the cops. However, the cops are not aware of the exact location of the robber while the robber is aware of every move they have made. Thus, the robber has perfect information in the localization game, but the cops do not, which at first sight contradicts our claim. Therefore, we propose the following reformulation of the game, which is based on a purely information theoretical perspective. When the cops put their sensors on the vertices of the set S_1, we partition the vertex set V into $R^1_1 \cup R^1_2 \cup \ldots \cup R^1_{\ell_1}$ where the sets $(R^1_j)_{1 \leq j \leq \ell_1}$ are the equivalence classes of vertices in V that have the same S_1-signature. Then, instead of choosing a specific location, the robber can choose some equivalence class $R^1_{j_1}$. Once the cops choose S_2, we partition the set $N[R^1_{j_1}]$ into equivalence classes $R^2_1 \cup R^2_2 \cup \ldots \cup R^2_{\ell_2}$ so that every vertex in R^2_j has the same S_2-signature. Then, the robber chooses a set among $(R^2_{j_2})_{1 \leq j_2 \leq \ell_2}$. Iteratively, in round i, once the cops choose S_i, this gives the partition $N[R^{i-1}_{j_{i-1}}] = R^i_1 \cup R^i_2 \cup \ldots \cup R^i_{\ell_i}$ with every vertex in R^i_j having the same S_i-signature; then the robber chooses some $R^i_{j_i}$. In this version of the game, the cops win in round i if the robber is forced to choose a set $R^i_{j_i}$ with only one vertex, that is, $|R^i_{j_i}| = 1$. In this reformulation, both players have perfect information. In particular, the localization game is a combinatorial game and so one of the players must have a winning strategy, that is, a strategy which wins against all of the other player's strategies simultaneously. We direct the reader to [9] for a longer discussion.

2.2 Notation

Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined by $(0, x) \sim (\sqrt{n}, x)$ and $(x, 0) \sim (x, \sqrt{n})$ for every $x \in \mathbb{R}$. The *torus* T_n is defined as $T_n = \mathbb{R}^2/\sim$ and is equipped with the natural metric d_{T_n}, inherited from the Euclidean metric d_E on $\mathbb{R}^2$. To simplify notation, we write d_T instead of d_{T_n} below. The following definitions are used for both the Euclidean distance as well as the distance on the torus. For a given $x \in [0, \sqrt{n}]^2$ (respectively, $x \in T_n$) and $r \geq 0$, let $\mathcal{B}(x, r)$ be the *(closed) ball with center x and radius r*, that is, $\mathcal{B}(x, r) = \{y \in [0, \sqrt{n}]^2 : d_E(x, y) \leq r\}$ (on T_n we have $\mathcal{B}(x, r) = \{y \in T_n : d_T(x, y) \leq r\}$). Let $\mathcal{C}(x, r)$ be the *circle with center x and radius r*, that is, $\mathcal{C}(x, r) = \{y \in [0, \sqrt{n}]^2 : d_E(x, y) = r\}$ (again, on T_n we have $\mathcal{C}(x, r) = \{y \in T_n : d_T(x, y) = r\}$). Finally, for $0 \leq r_1 \leq r_2$, let $\mathcal{D}(x, r_1, r_2) = \mathcal{B}(x, r_2) \setminus \mathcal{B}(x, r_1)$ be the *crown with center x and radii r_1 and r_2*. For any $d \geq 0$, we also use the term *strip of width d* to denote the set of points in $\mathbb{R}^2$ at distance at most $d/2$ from a fixed line.

As typical in the field of random graphs, we will use $\log x$ to denote the natural logarithm of x. Finally, for expressions that clearly have to be integer valued, we systematically round up or down without specifying which since the choice does not affect our arguments.

2.3 De-Poissonization

In order to simplify some of our proofs, we will make use of a technique known as *de-Poissonization*, which has many applications in geometric probability (see [17] for a detailed account of the subject). Here we only roughly sketch the idea behind it.

Consider the following related models of random geometric graphs. Let $V = V'$, where V' is a set given by a homogeneous Poisson point process of intensity 1 in $[0, \sqrt{n}]^2$, respectively in T_n. In other words, V' consists of N points in the square $[0, \sqrt{n}]^2$, or in the torus T_n, chosen independently and uniformly at random, where N is a Poisson random variable with expectation equal to n. By analogy to the models $\mathcal{G}(n, r)$ and $\mathcal{T}(n, r)$, almost surely no two vertices are located at the same position, and we are therefore allowed to identify any vertex v_i with its geometric position x_i in $[0, \sqrt{n}]^2$, respectively in T_n. Fix a parameter $r \geq 0$ and, for any pair of vertices u and v in V', connect u and v if $d_E(u, v) \leq r$, when working with $\mathcal{G}(N, r)$, and if $d_T(u, v) \leq r$, when working with $\mathcal{T}(N, r)$. We denote these new models by $\mathcal{G}_{Po}(n, r)$ and $\mathcal{T}_{Po}(n, r)$.

Since our main result deals with the $\mathcal{T}(n, r)$ model, we concentrate on the connection between the models $\mathcal{T}(n, r)$ and $\mathcal{T}_{Po}(n, r)$. The same relationship holds for $\mathcal{G}(n, r)$ and $\mathcal{G}_{Po}(n, r)$. The main advantage of defining V' via a Poisson point process is motivated by the following two properties: first, the number of vertices of V' that lie in any measurable set $A \subseteq T_n$ of Lebesgue measure a has a Poisson distribution with expectation a, and second, the number of vertices of V' in disjoint subsets of T_n are independently distributed. Moreover, by conditioning $\mathcal{T}_{Po}(n, r)$ upon the event $N = n$, we recover the original distribution of $\mathcal{T}(n, r)$. Therefore, since $\mathbb{P}(N = n) = \Theta(1/\sqrt{n})$, any event holding in $\mathcal{T}_{Po}(n, r)$ with probability at least $1 - o(f_n)$ must hold in $\mathcal{T}(n, r)$ with probability at least $1 - o(f_n\sqrt{n})$.

We may also transfer results that hold in $\mathcal{T}(n, r)$ to $\mathcal{T}_{Po}(n, r)$. For example, suppose that for some random variable $X = X(G)$, there exist non-decreasing functions $f(n)$ and $g(n)$ such that a.a.s. $f(n) \leq X \leq g(n)$ for $G \in \mathcal{T}(n, r)$. Then, since a.a.s. $(1 - \varepsilon)n \leq N \leq (1 + \varepsilon)n$ for some $\varepsilon = \varepsilon(n) = o(1)$, we get that a.a.s. $f((1 - \varepsilon)n) \leq X \leq g((1 + \varepsilon)n)$ for $G \in \mathcal{T}_{Po}(n, r)$. In particular, our main result, Theorem 1.1, holds for $G \in \mathcal{T}_{Po}(n, r)$ as well.

2.4 Concentration inequalities

Let us first state a few specific instances of Chernoff's bound that we will find useful. Let $X \sim \mathrm{Bin}(n, p)$ be a random variable distributed according to a Binomial distribution with parameters n and p. Then, a consequence of *Chernoff's bound* (see e.g. [13, Theorem 2.1]) is that for any $t \geq 0$ we have

$$\mathbb{P}(X \geq \mathbb{E}X + t) \leq \exp\left(-\frac{t^2}{2(\mathbb{E}X + t/3)}\right) \quad (1)$$

$$\mathbb{P}(X \leq \mathbb{E}X - t) \leq \exp\left(-\frac{t^2}{2\mathbb{E}X}\right). \quad (2)$$

Moreover, let us mention that the bound holds in a more general setting as well, that is, for $X = \sum_{i=1}^{n} X_i$ where $(X_i)_{1\le i\le n}$ are independent variables and for every $i \in \{1, 2, \ldots, n\}$ we have $X_i \sim \text{Bernoulli}(p_i)$ with (possibly) different p_i-s (again, see e.g. [13] for more details).

We will also need the following generalization of the previous bound due to Bentkus [1], stated in a simplified form here. For two random variables X and Y defined on the same probability space, we write $X \preccurlyeq Y$ if Y stochastically dominates X, that is, $\mathbb{P}(X \ge x) \le \mathbb{P}(Y \ge x)$ for all $x \in \mathbb{R}$. Let $\mathcal{L}(X)$ denote the distribution of the random variable X. For a positive random variable Y and for $m > 0$ we define the random variable $Y^{[m]}$ so that $\mathbb{E}Y^{[m]} = m$, $Y^{[m]} \preccurlyeq Y$ and so that for some $b > 0$ we have $\mathbb{P}(0 < Y^{[m]} < b) = 0$ and $\mathbb{P}(Y^{[m]} \ge x) = \mathbb{P}(Y \ge x)$ for all $x \ge b$ (in other words, one may roughly think of $Y^{[m]}$ as the random variable "shifting mass that is close to 0 to 0 itself").

Lemma 2.1 ([1])**.** *Let $S = X_1 + \ldots + X_\ell$ be a sum of ℓ positive independent random variables. Assume that for every $k \in \{1, 2, \ldots, \ell\}$ we have $X_k \preccurlyeq Y$ and $\mathbb{E}X_k \le m$ for some positive random variable Y and some non-negative real number $m \le \mathbb{E}Y$. Let $T = \varepsilon_1 + \ldots + \varepsilon_\ell$ be a sum of ℓ independent random variables ε_k so that $\mathcal{L}(\varepsilon_k) = \mathcal{L}(Y^{[m]})$. Then, for all $x \in \mathbb{R}$,*

$$\mathbb{P}(S \ge x) \le \inf_{h \le x} e^{-hx}\mathbb{E}e^{hT}.$$

In particular, if $\mathbb{E}S \ge 1/c$ for some constant $c > 0$,

$$\mathbb{P}(S \ge c\,\mathbb{E}S) \le e^{-c\mathbb{E}S}\mathbb{E}e^{T}.$$

2.5 Euclidean vs. graph distances

Let us start with the following result from [8].

Theorem 2.2 ([8], Theorem 1.1 (ii))**.** *Fix $r = r(n) \ge r_0$ and let $G \in \mathcal{G}(n, r)$. Then, a.a.s. for all pairs of vertices $u, v \in V(G)$ we have $d_G(u, v) \le \left\lceil \frac{d_E(u,v)}{r}(1 + \gamma r^{-4/3}) \right\rceil$, where*

$$\gamma = \max\left(31\left(\frac{2r\log n}{r + d_E(u,v)}\right)^{2/3}, \frac{70\log^2 n}{r^{8/3}}, 300^{2/3}\right). \tag{3}$$

As we plan to investigate $\mathcal{T}(n, r)$ instead of $\mathcal{G}(n, r)$, we need to adapt the above theorem to the torus metric. Fortunately, the adjustment is straightforward.

Corollary 2.3. *Fix $r = r(n) \ge r_0$ and let $G \in \mathcal{T}(n, r)$. Then, a.a.s. the following property holds for all pairs of vertices $u, v \in V(G)$:*

$$d_G(u, v) \le \left\lceil \frac{d_T(u, v)}{r}(1 + \gamma r^{-4/3}) \right\rceil,$$

where γ is defined in (3).

Proof. We generate $\mathcal{T}(n, r)$ and nine copies of $\mathcal{G}(n, r)$ that will be coupled in the following way. Start with n random points $x_1, x_2, \ldots, x_n$ in the square $[0, \sqrt{n}]^2$ sampled independently and uniformly. We use these points to generate $G \in \mathcal{T}(n, r)$. We stay with these n points on the torus, and then translate our $\sqrt{n} \times \sqrt{n}$-window by the vector $(i\sqrt{n}/2, j\sqrt{n}/2)$ for some $i, j \in \{-1, 0, 1\}$; in other words, we consider the square $[i\sqrt{n}/2, \sqrt{n} + i\sqrt{n}/2] \times [j\sqrt{n}/2, \sqrt{n} + j\sqrt{n}/2]$. In fact, for example, the squares corresponding to $(i, j) = (-1, -1)$ and to $(i, j) = (1, 1)$ coincide but it will be convenient to keep 9 squares instead of 4. Indeed, for any two points u, v in the original square, the toroidal distance between u and v is the minimum distance between u (taken in the original square) and all 9 images of the vertex v under the above translations. Each of these 9 choices yields one copy of $G_{ij} \in \mathcal{G}(n, r)$.

Since we aim for a statement that holds a.a.s., we may assume that for each G_{ij}, the statement of Theorem 2.2 is satisfied. Since we have 10 graphs and 9 squares (one graph for each of the images of the square $[0, \sqrt{n}]^2$ under the above translations, and the graph on T_n), we will use superscripts to indicate which graph/square we consider. Consider any pair of vertices $u, v \in V(G)$. Clearly, for some G_{ij} we have $d_T(u,v) = d_E^{G_{ij}}(u,v)$. Indeed, the shortest segment uv in T_n is contained in some square of side length $\sqrt{n}/2$, and any such square is contained in some of the nine squares $([i\sqrt{n}/2, \sqrt{n}+i\sqrt{n}/2] \times [j\sqrt{n}/2, \sqrt{n}+j\sqrt{n}/2])_{i,j\in\{-1,0,1\}}$. Also, since G_{ij} is a subgraph of G, $d_G^G(u,v) \le d_G^{G_{ij}}(u,v)$. Combining these observations together we get that

$$d_G^G(u,v) \le d_G^{G_{ij}}(u,v) \le \left\lceil \frac{d_E^{G_{ij}}(u,v)}{r}(1+\gamma r^{-4/3}) \right\rceil = \left\lceil \frac{d_T(u,v)}{r}(1+\gamma r^{-4/3}) \right\rceil .$$

The proof of the corollary is finished. □

We will also need the following simple but useful observation.

Observation 2.4. *Let $G \in \mathcal{T}(n,r)$. Then, a.a.s. for any point $x \in T_n$ there exists a vertex $v_i \in V(G)$ such that $d_T(x, v_i) \le 2\sqrt{\log n}$.*

Proof. Fix $k = \lfloor \sqrt{n/\log n}/1.1 \rfloor$. Tessellate T_n into k^2 small squares, each of side length $\sqrt{n}/k = (1.1 + o(1))\sqrt{\log n}$. The probability that a given small square contains no vertex is equal to

$$\left(1 - \frac{(\sqrt{n}/k)^2}{n}\right)^n \le \exp\Big(-(\sqrt{n}/k)^2\Big) = \exp\Big(-(1.21+o(1))\log n\Big) = o(n^{-1}).$$

Since there are $k^2 = o(n)$ small squares, it follows from the union bound over all small squares that a.a.s. each small square contains a vertex. Since we aim for a statement that holds a.a.s., we may assume that this property holds and then the conclusion follows deterministically. Indeed, since $1.1\sqrt{2} < 2$, for any point $x \in T_n$ the ball $\mathcal{B}(x, 2\sqrt{\log n})$ contains at least one square, which implies the result. □

3 Upper bound

This section is devoted to the proof of the upper bounds stated in Theorem 1.1.

Let us start by showing that the cops are able to localize the robber within a square of side length $20000\,r$ by using only four sensors. We prepare the ground with the following lemma.

Lemma 3.1. *Fix $r = r(n) \ge r_0$ and let $G \in \mathcal{T}(n,r)$. Suppose that $\mathcal{T}(n,r)$ satisfies the properties stated in Corollary 2.3 and Observation 2.4. Let $s = s(n)$ be such that $20000\,r \le s \le \sqrt{n}/9$. Suppose that at the beginning of some round the robber occupies a vertex inside a square S of side length s and at Euclidean distance at least r from the border of S. Then, the cops may place four sensors so that at the end of the current round the robber is localized within a square of side length $s/4$ and at distance at least r from the border of this square.*

Proof. Consider four points $A, B, C, D \in T_n$ that are the four corners of the square S' of side length $3s$, with sides parallel to the sides of S, and containing the square S in its center. By our assumption, $\max_{u,v\in S'} d_T(u,v) \le 3\sqrt{2}s \le \sqrt{2n}/3 < \sqrt{n}/2$ (note that S is not necessarily axis-parallel); in particular, the geodesic between any two points in S' is included in S', that is, S' is small enough so that the metric d_T on T_n coincides with the Euclidean metric on the square S'. Place sensors at the vertices v_A, v_B, v_C, v_D that are the closest to A, B, C, D (not necessarily in the square $ABCD$), respectively. By Observation 2.4, we may find a vertex within Euclidean distance $2\sqrt{\log n}$ for any choice of points A, B, C, D. Let d_A, d_B, d_C, d_D

be the graph distances from sensors v_A, v_B, v_C, v_D, respectively, to the robber once she makes her move (that is, after step b) of the current round). By our assumption, she is still inside the square S.

Now, by Corollary 2.3 and the fact that for all $i \in \{A, B, C, D\}$ we have $rd_G(v_i, R) \geq d_T(v_i, R) = d_E(v_i, R)$, the robber is in the crown

$$\mathcal{D}\left(v_i, \frac{r(d_i-1)}{1+\gamma r^{-4/3}}, rd_i\right) = \mathcal{B}(v_i, rd_i) \setminus \mathcal{B}\left(v_i, \frac{r(d_i-1)}{1+\gamma r^{-4/3}}\right).$$

Note that the Euclidean distance between any point $i \in \{A, B, C, D\}$ and the position of the robber after she moves is at least $d_E(i, S) = \sqrt{2}s$ (see Figure 1). Hence, since $r \geq r_0$ and $s \geq 20000\, r$, we get that for every $i \in \{A, B, C, D\}$,

$$d_i \geq \left\lceil \frac{\sqrt{2}s - 2\sqrt{\log n}}{r} \right\rceil \geq 20000.$$

Moreover, $\gamma r^{-4/3} \leq 1/50$: indeed, we have

$$\begin{aligned} 31\left(\frac{2r\log n}{r+d_E(u,v)}\right)^{2/3} r^{-4/3} &\leq 31\left(\frac{2r\log n}{20000r}\right)^{2/3} 70^{-4/3}\log^{-2/3} n = \frac{31}{10^{8/3}70^{4/3}} < \frac{1}{50}, \\ \frac{70\log^2 n}{r^{8/3}} r^{-4/3} &= \frac{70\log^2 n}{r^4} \leq \frac{70}{70^4} < \frac{1}{50}, \text{ and} \\ 300^{2/3} r^{-4/3} &\ll \frac{1}{50}. \end{aligned}$$

Therefore, each of the four crowns

$$\left(\mathcal{D}\left(v_i, \frac{r(d_i-1)}{1+\gamma r^{-4/3}}, rd_i\right)\right)_{1\leq i\leq 4}$$

has width

$$rd_i - \frac{r(d_i-1)}{1+\gamma r^{-4/3}} \leq rd_i/51 + 50r/51 \leq (3/50 + 1/20000)s \leq s/6 - 4\sqrt{\log n}.$$

The first and the third inequalities follow from a direct computation, while the second inequality uses the fact that

$$rd_i \leq \frac{51}{50} d_E(v_i, R) + r \leq \frac{51}{50}(d_E(i, R) + 2\sqrt{\log n}) + r \leq \frac{51}{50}(2\sqrt{2}s + r) + r \leq \frac{51}{50} \cdot 3s.$$

Since for every $i \in \{A, B, C, D\}$ we have $d_E(i, v_i) \leq 2\sqrt{\log n}$, we get that for any radius $\rho \geq 0$ we have

$$\mathcal{D}(v_i, \rho + 2\sqrt{\log n}, \rho + s/6 - 2\sqrt{\log n}) \subseteq \mathcal{D}(i, \rho, \rho + s/6),$$

so the robber must be hiding inside

$$\bigcap_{i\in\{A,B,C,D\}} \mathcal{D}(v_i, \rho_i + 2\sqrt{\log n}, \rho_i + s/6 - 2\sqrt{\log n}) \subseteq \bigcap_{i\in\{A,B,C,D\}} \mathcal{D}(i, \rho_i, \rho_i + s/6),$$

where

$$\rho_i = \frac{r(d_i-1)}{1+\gamma r^{-4/3}} - 2\sqrt{\log n}. \tag{4}$$

It remains to show that the four crowns with centers at the corners of the square $ABCD$ intersect in a region, which is contained in a square of side at most $s/4 - 2r$. The next purely geometric claim is the key to our proof of this fact.

Claim 3.2. *Let $\rho_B, \rho_D > 0$ be such that*

$$\mathcal{D}(B, \rho_B, \rho_B + s/6) \cap \mathcal{D}(D, \rho_D, \rho_D + s/6) \cap S \neq \emptyset.$$

Then, $\mathcal{D}(B, \rho_B, \rho_B + s/6) \cap \mathcal{D}(D, \rho_D, \rho_D + s/6)$ is included in a strip, parallel to the diagonal AC and of width at most $s/4 - 2r$.

Proof. If $\rho_B + \rho_D + s/6 \leq |BD|$, then one may easily conclude that

$$\mathcal{D}(B, \rho_B, \rho_B + s/6) \cap \mathcal{D}(D, \rho_D, \rho_D + s/6)$$

is included in a strip between two lines, parallel to AC and at distance at most $s/6 \leq s/4 - 2r$.

Otherwise, let $\mathcal{C}(B, \rho_B) \cap \mathcal{C}(D, \rho_D + s/6) = \{Q', Q''\}$ with Q', A on the same side with respect to BD, $\mathcal{C}(B, \rho_B + s/6) \cap \mathcal{C}(D, \rho_D + s/6) = \{P', P''\}$ with P', A on the same side with respect to BD, and $\mathcal{C}(B, \rho_B + s/6) \cap \mathcal{C}(D, \rho_D) = \{R', R''\}$ with R', A on the same side with respect to BD. We know that $Q'Q'' \parallel P'P'' \parallel R'R''$, and the three of them are parallel to AC. Also, define P, Q, and R as the intersection points of BD with the segments $P'P'', Q'Q''$ and $R'R''$, respectively. See Figure 1 for an illustration. By the Pythagorean theorem

$$|DP|^2 - |PB|^2 = |DP''|^2 - |P''B|^2 = (\rho_D + s/6)^2 - (\rho_B + s/6)^2,$$

and

$$|DQ|^2 - |QB|^2 = |DQ''|^2 - |Q''B|^2 = (\rho_D + s/6)^2 - \rho_B^2.$$

We conclude that

$$\begin{aligned} s^2/36 + \rho_B s/3 &= (|DQ|^2 - |QB|^2) - (|DP|^2 - |PB|^2) \\ &= (|DQ| - |QB|)|DB| - (|DP| - |PB|)|DB| \\ &= 2 \cdot |PQ| \cdot |DB| = 6\sqrt{2}s \cdot |PQ|. \end{aligned}$$

Since $\rho_B \leq (2\sqrt{2} + 1/6)s$ (recall that $\mathcal{D}(B, \rho_B, \rho_B + s/6) \cap S \neq \emptyset$), we have that

$$|PQ| \leq \frac{1}{6\sqrt{2}} \left(\frac{2\sqrt{2}}{3} + \frac{1}{6 \cdot 3} + \frac{1}{36} \right) s < 0.121 \cdot s.$$

A similar argument implies that

$$|QR| \leq \frac{1}{6\sqrt{2}} \left(\frac{2\sqrt{2}}{3} + \frac{1}{6 \cdot 3} + \frac{1}{36} \right) s < 0.121 \cdot s.$$

Thus, the strip between the lines $Q'Q''$ and $R'R''$ contains $\mathcal{D}(B, \rho_B, \rho_B + s/6) \cap \mathcal{D}(D, \rho_D, \rho_D + s/6)$, and the distance between these two lines is given by $|RQ| = |RP| + |PQ| < 2 \cdot 0.121 \cdot s = 0.242 \cdot s < s/4 - 2r$. The proof of the claim is finished. □

Applying Claim 3.2 to the intersection

$$\mathcal{D}\left(A, \rho_A, \rho_A + s/6\right) \cap \mathcal{D}\left(C, \rho_C, \rho_C + s/6\right),$$

and to the intersection

$$\mathcal{D}\left(B, \rho_B, \rho_B + s/6\right) \cap \mathcal{D}\left(D, \rho_D, \rho_D + s/6\right),$$

we get that the intersection of all these four crowns is contained in a square of side $s/4 - 2r$. This square is situated in the center of a larger square with sides parallel to the sides of the smaller square, and of length $s/4$, which finishes the proof of the lemma. □

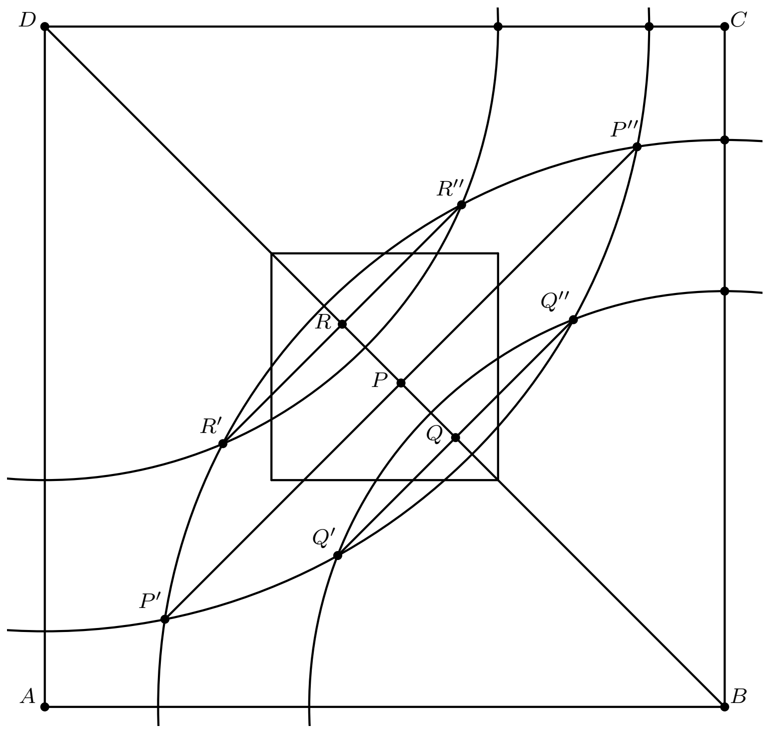

Figure 1: Illustration for the proof of Claim 3.2.

Now, we put all observations together and show that, throughout several rounds, the cops are able to localize the robber within a square of side length $20000\, r$ by using only four sensors.

Corollary 3.3. *Using only four sensors, a.a.s. the robber can be localized on $\mathcal{T}(n, r)$ within a square of side length $20000\, r$.*

Proof. If $r \geq \sqrt{n}/20000$, then there is nothing to prove. Hence, suppose that $r \leq \sqrt{n}/20000$. Since we aim for a statement that holds a.a.s., we may assume that $\mathcal{T}(n, r)$ satisfies the properties stated in Corollary 2.3 and Observation 2.4.

Let B be the set of the 400 vertices of the 20×20 square grid of mesh size $\sqrt{n}/20$, covering T_n. Construct a set C by adding, for every vertex h of B, a vertex of G at distance at most $2\sqrt{\log n}$ from h (the existence of such vertex is guaranteed by Observation 2.4; if there are more choices for a given h, then we may choose arbitrarily). Let the cops put sensors at the vertices of C in groups of 4, one group after another, so that all vertices are tested in 100 rounds. Trivially, from the first test to the last one, the robber changes her position by at most $100r$.

Let $u \in C$ be a vertex that detected the closest graph distance to the robber (if there are many such vertices in C, then we select one of them arbitrarily). Our goal is to estimate the Euclidean distance (coinciding with the distance on the torus T_n) from u to the robber once testing is finished (that is, after 100 initial rounds). Note that the robber had to be initially at distance at most $\sqrt{2n}/40$ from some point in B and so at distance at most $\sqrt{2n}/40 + 2\sqrt{\log n} = (\sqrt{2}/40 + o(1))\sqrt{n}$ from some vertex w in C. Hence, she is certainly at distance at most $(\sqrt{2}/40 + o(1))\sqrt{n} + 100r < \sqrt{n}/24$ from w when w was probed. More importantly, by Corollary 2.3 (and the computations done in the proof of Lemma 3.1), we know that at that point of the game the graph distance from w to the robber was at most

$$\frac{51}{50} \cdot \frac{\sqrt{n}}{24} \cdot \frac{1}{r} + 1.$$

As a result, since u is the sensor that returned the smallest graph distance, the graph distance from u to the position she was when u was probed is at most

$$\frac{51}{50} \cdot \frac{\sqrt{n}}{24} \cdot \frac{1}{r} + 1,$$

and so the Euclidean distance between u and the position of the robber at the end of round 100 is at most

$$\frac{51}{50} \cdot \frac{\sqrt{n}}{24} + r + 100r < \sqrt{n}/20.$$

Hence, the cops have a strategy to find a square of side length $\sqrt{n}/10$ in which the robber is located at the end of round 100. By making the square slightly larger (that is, of side length $\sqrt{n}/9$), we are guaranteed that she is at distance at least r from the border. Finally, we may consecutively apply Lemma 3.1 to get the desired property and finish the proof. □

At this point of the game, we may assume that the robber occupies a vertex in a region $\mathcal{R}$ that is inside a square of side length $20000\, r$. For a region $\mathcal{R}$, we define $N[\mathcal{R}] \subseteq V(G)$ as the subset of the vertices of G contained in the union of all balls of radius r, centered at the vertices in $V(G) \cap \mathcal{R}$. The cops aim to finish the game in the very next round by choosing a set W of vertices to put sensors on such that, regardless where the robber moves, she is going to be localized. In other words, their goal is to partition $N[\mathcal{R}]$ into equivalence classes with the same W-signature such that each class consists of a single vertex (see Subsection 2.1 for a convenient reformulation of the game that explains this line of thinking). In this case, we also say that the set of sensors W *distinguishes* the vertices in the set $N[\mathcal{R}] \subseteq V(G)$. Trivially, $N[\mathcal{R}]$ is contained in a square S of side length $20002\, r$. Of course, the robber plays the game optimally so she can try to "get trapped" in a region $\mathcal{R}$ that is placed in some convenient (for her) part of the square $[0, \sqrt{n}]^2$. Hence, we need to show that, regardless what she does, she will suffer the same fate and lose the game in the very next round.

Let $\mathcal{F}$ be a family of squares of side length $10^5\, r$, with sides parallel to the axes, and with left-bottom vertices at points $(10^4\, r\, i, 10^4\, r\, j)$ for some $i, j \in \mathbb{N} \cup \{0\}$ such that $10^4\, r\, i < \sqrt{n}$ and $10^4\, r\, j < \sqrt{n}$. Clearly, $|\mathcal{F}| = O(n/r^2) < n$. For a given square $S \in \mathcal{F}$, let $I(S)$ be defined as the square of side length $10^5 r - 2r$ inside S, centered at the same point as S and with sides, parallel to the sides of S. We call $I(S)$ the *internal square* of S. Clearly, $N[\mathcal{R}] \subseteq I(S) \subseteq S$ for some $S \in \mathcal{F}$. Hence, in order to finish the proof of the upper bound, it remains to show the following lemma.

Lemma 3.4. *Fix $r = r(n) \in [r_0, \sqrt{n}/4)$ and let $G \in \mathcal{T}(n, r)$. Let*

$$w = w(n) = \begin{cases} 10^{15}\, r^{4/3} & \text{if } r \geq \log^{3/2} n, \\ 3 \cdot 10^{16} \log^2 n & \text{if } 100 \log n \leq r < \log^{3/2} n, \\ 2 \cdot 10^{10} r^2 & \text{if } r < 100 \log n. \end{cases}$$

Then, a.a.s. the following property holds: for any square $S \in \mathcal{F}$, there exists a set of vertices $W = W(S) \subseteq S \cap V(G)$ of cardinality at most w such that placing sensors on W distinguishes all vertices in the internal square $I(S)$, that is, all vertices in $I(S)$ have a unique W-signature.

Before diving into the proof of Lemma 3.4, we provide a rough sketch of the main idea. Let us fix $\delta = \delta(n) = o(1)$ (to be chosen appropriately later on). For every square $S \in \mathcal{F}$, the set $W = W(S)$ is constructed as follows: we investigate all vertices in S and independently put them into a set X with probability δ. This set partitions the vertices in the internal square $I(S)$ into equivalence classes with the same X-signature. We do not expect each class to contain only one vertex so we investigate all equivalence classes. If some class contains at least two vertices, then we put all vertices from that class into a set Y. (In fact, we may put all but one of them into Y but, for simplicity, we include all of them as it would not improve the asymptotic order of the bound.) By construction, the set $W = X \cup Y$ achieves the desired

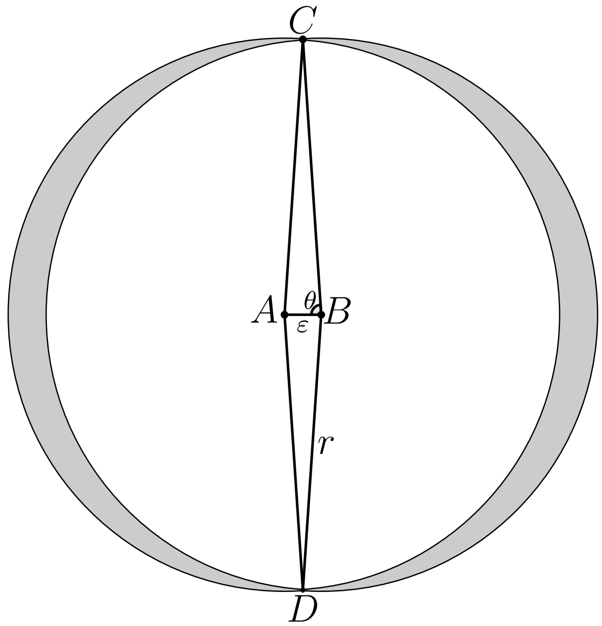

Figure 2: An illustration from the computation in (5).

goal of identifying the robber since each non-sensor vertex in $I(S)$ has a unique X-signature (otherwise, it would be put into Y) and so also a unique W-signature, and each sensor vertex in $I(S)$ has a unique W-signature as it is the only vertex at distance 0 from itself. Note that, roughly speaking, if X is small, then Y has to be large and vice versa. Hence, at some point we will have to optimize δ as a function of r since we aim to find a set W, which is as small as possible.

In the next observation we investigate $\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)$, the symmetric difference of two discs centered in A and B. We show a lower bound for the area of this symmetric difference which is a non-decreasing function of the distance between A and B.

Observation 3.5. *Fix $r = r(n) < \sqrt{n}/4$ and let A, B be any two points in T_n at distance ε from each other. If $\varepsilon \le \varepsilon_0 := 2r$, then*

$$|\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)| = (2\pi - 4\arccos(\varepsilon/2r))r^2 + 2\varepsilon r\sqrt{1-\frac{\varepsilon^2}{4r^2}} \ge 2\varepsilon r.$$

In particular, if $\varepsilon \ll r$, then

$$|\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)| = (4+o(1))\varepsilon r.$$

On the other hand, if $\varepsilon > \varepsilon_0$, then trivially

$$|\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)| = |\mathcal{B}(A,r)| + |\mathcal{B}(B,r)| = 2\pi r^2 \ge 2\varepsilon_0 r.$$

Proof. Since $r < \sqrt{n}/4$ we have that $\mathcal{B}(A,r)\cap\mathcal{B}(B,r)$ is a (possibly empty) connected subset of T_n. Let $\mathcal{C}(A,r)\cap\mathcal{C}(B,r) = \{C,D\}$, $\angle CBA = \angle ABD = \theta$, and let ε be the distance between A and B. Suppose that $\varepsilon \le 2r$ since the statement for larger values of ε is trivial. (An illustration of the configuration may be found on Figure 2.) The area of $\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)$ (the grey region in Figure 2) is by a simple inclusion-exclusion formula equal to

$$2\,\frac{2\pi-2\theta}{2\pi}\pi r^2 - 2\,\frac{2\theta}{2\pi}\pi r^2 + 2\cdot 2\,\frac{\varepsilon r\sin\theta}{2} = \left(2\pi - 4\arccos\left(\frac{\varepsilon}{2r}\right)\right)r^2 + 2\varepsilon r\sqrt{1-\frac{\varepsilon^2}{4r^2}}.$$

The desired bound holds since $\arccos(\varepsilon/2r) \le \pi/2 - \varepsilon/2r$.

If we additionally suppose that $\varepsilon \ll r$, then $\sin(\theta) = 1 - o(1)$ and the above equality can be simplified as follows:

$$2\,\frac{2\pi - 2\theta}{2\pi}\pi r^2 - 2\,\frac{2\theta}{2\pi}\pi r^2 + (4+o(1))\frac{\varepsilon r}{2} = (2\pi - 4\theta)r^2 + (2+o(1))\varepsilon r. \tag{5}$$

Moreover, $\cos(\theta) = \varepsilon/2r = o(1)$ and thus

$$\theta = \arccos(\varepsilon/2r) = \frac{\pi}{2} - \frac{\varepsilon}{2r} - O\left(\frac{\varepsilon^3}{8r^3}\right) = \frac{\pi}{2} - (1+o(1))\frac{\varepsilon}{2r},$$

and so

$$|\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)| = (2\pi - 4\theta)r^2 + (2+o(1))\varepsilon r = (4+o(1))\varepsilon r.$$

The proof of the observation is finished. □

Observation 3.5 is enough to show that, for any two points A and B on the torus T_n, with high probability there are many vertices of $G \in \mathcal{T}(n,r)$ in the symmetric difference of $\mathcal{B}(A,r)$ and $\mathcal{B}(B,r)$ provided that A and B are "sufficiently far from each other".

Lemma 3.6. *Fix $r = r(n) \in [r_0, \sqrt{n}/4)$ and let $G \in \mathcal{T}(n,r)$. Let*

$$\varepsilon_c = \begin{cases} 12\, r^{-1/3}, & \text{if } r \ge \log^{3/2} n, \text{ and} \\ 12 \log n / r, & \text{otherwise.} \end{cases}$$

Then a.a.s. the following property holds: for any pair of vertices of G with positions A, B such that $d_T(A,B) = \varepsilon \ge \varepsilon_c$, the number of vertices in $\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)$ is at least $\min(\varepsilon, 2r)r$.

Proof. Consider the positions A, B of *any* two vertices of G at distance $\varepsilon \ge \varepsilon_c$ from each other. By Observation 3.5, $|\mathcal{B}(A,r)\triangle\mathcal{B}(B,r)| \ge a := 2\min(\varepsilon, 2r)r$. Hence, the number of vertices in the symmetric difference can be stochastically bounded from below by a random variable $X \sim \mathrm{Bin}(n-2, a/n)$ with $\mathbb{E}X = (n-2)a/n = (1+o(1))a$. Note that if $r \ge \log^{3/2} n$, then $a \ge 2\varepsilon_c r = 24 r^{2/3} \ge 24 \log n$; otherwise, since $\varepsilon_c \le 12 \log n / r_0 \le 2r_0 \le 2r$, it is also true that $a \ge 2\varepsilon_c r = 24 \log n$. In either case, it follows from the Chernoff's bound (2), applied with $t = \mathbb{E}X - a/2 = (1+o(1))a/2$, that

$$\begin{aligned} \mathbb{P}(X \le a/2) &= \mathbb{P}(X \le \mathbb{E}X - t) \le \exp\left(-\frac{t^2}{2\mathbb{E}X}\right) = \exp\left(-\left(\frac{1}{8} + o(1)\right)a\right) \\ &\le \exp\left(-(3+o(1))\log n\right) = o\left(\frac{1}{n^2}\right). \end{aligned}$$

The lemma holds by a union bound over all pairs of vertices. □

The next lemma controls the number of pairs of vertices at a given distance from each other.

Lemma 3.7. *Fix $r = r(n) \in [r_0, \sqrt{n}/4)$ and let $G \in \mathcal{T}(n,r)$. Then, a.a.s. the following properties hold for all squares $S \in \mathcal{F}$.*

(a) The number of vertices in S is at most $2 \cdot 10^{10} r^2$.

(b) Let $\varepsilon_c = 12\, r^{-1/3}$. If $r \ge \log^{3/2} n$, then for any given $k = k(n) \in \mathbb{N} \cup \{0\}$ satisfying $\varepsilon = \varepsilon(k) := 2^k \varepsilon_c \le r^{-0.1}$, the number of pairs of vertices in $I(S)$ that are at distance at most ε from each other is at most $2 \cdot 10^{12} r^2 \varepsilon^2$.

(c) Let $\varepsilon_c = 12 \log n / r$. If $\log^{5/4} n \le r < \log^{3/2} n$, then for any given $k = k(n) \in \mathbb{N} \cup \{0\}$ satisfying $\varepsilon = \varepsilon(k) := 2^k \varepsilon_c \le r^{-0.1}$, the number of pairs of vertices in $I(S)$ that are at distance at most ε from each other is at most $2 \cdot 10^{12} r^2 \varepsilon^2$.

(d) Let $\varepsilon_c = 12 \log n/r$. If $100 \log n \le r < \log^{5/4} n$, then the number of pairs of vertices in $I(S)$ that are at distance at most ε_c from each other is at most $10^{16} \log^2 n$.

Proof. We prove part (a) first. Let us concentrate on any square $S \in \mathcal{F}$. Recall that the area of S is $(10^5 r)^2 = 10^{10} r^2$. Hence, the number of vertices in S is equal to $X \sim \mathrm{Bin}(n, 10^{10}r^2/n)$ with $\mathbb{E}X = 10^{10}r^2 \ge 49 \cdot 10^{12} \log n$. It follows immediately from Chernoff's bound (1) that

$$\begin{aligned}\mathbb{P}(X \ge 2 \cdot 10^{10} r^2) &= \mathbb{P}(X \ge \mathbb{E}X + \mathbb{E}X) \le \exp\left(-\frac{(\mathbb{E}X)^2}{2(\mathbb{E}X + \mathbb{E}X/3)}\right) \\ &= \exp\left(-\frac{3}{8}\mathbb{E}X\right) \le \exp\left(-10^{12}\log n\right) = o(1/n).\end{aligned}$$

Since the number of squares in $\mathcal{F}$ is less than n, the desired conclusion holds by a union bound over all squares.

In order to simplify the argument for part (b), we will use the de-Poissonization technique mentioned in Section 2.3. As before, let us concentrate on any square $S \in \mathcal{F}$. Without loss of generality, we may assume that the left-bottom corner of S is the point $(0,0)$ and the right-top corner of S is the point $(10^5 r, 10^5 r)$. Let us also fix $k = k(n) \in \mathbb{N} \cup \{0\}$ satisfying $\varepsilon = \varepsilon(k) := 2^k \varepsilon_c \le r^{-0.1}$.

For a given $a, b \in \{0,1\}$, let $\mathcal{E}_{a,b}$ be a family of *small* squares of side length 2ε, with sides parallel to the axes, and with left-bottom vertices at points $((a+2i)\varepsilon, (b+2j)\varepsilon)$ for some $i, j \in \mathbb{N} \cup \{0\}$ such that $(a+2i)\varepsilon < 10^5 r$ and $(b+2j)\varepsilon < 10^5 r$. Clearly, $|\mathcal{E}_{a,b}| = (1+o(1))10^{10}r^2/4\varepsilon^2 = O((r/\varepsilon_c)^2) = O(r^{8/3}) = O(n^{4/3})$. Moreover, any pair of vertices in $I(S)$ that are at distance at most ε from each other has to be included in some small square in some of the four families $\mathcal{E}_{a,b}$. Hence it is enough to bound from above the number of pairs of vertices contained in a small square in any of the four families $\mathcal{E}_{a,b}$.

Let us concentrate on one family $\mathcal{E}_{a,b}$ for a given $a, b \in \{0,1\}$. For any small square $s \in \mathcal{E}_{a,b}$, the number of vertices in s is equal to $\mathrm{Po}(\lambda)$ with $\lambda = (2\varepsilon)^2 = 4\varepsilon^2$. For $\ell \ge 2$, let $Z_s^{\ge \ell}$ be the random variable counting the number of vertices in s if this number is at least ℓ, and 0 otherwise. For $k \ge \ell$, we have

$$\mathbb{P}\left(Z_s^{\ge \ell} = k\right) = \mathbb{P}\left(\mathrm{Po}(\lambda) = k\right) = \frac{\lambda^k}{k!}\exp(-\lambda),$$

whereas for $0 < k < \ell$ the probability is 0 by definition. Since $\varepsilon \le r^{-0.1} = o(1)$ (and so also $\lambda = o(1)$),

$$\mathbb{E}Z_s^{\ge \ell} = \sum_{k \ge \ell} k \cdot \frac{\lambda^k}{k!}\exp(-\lambda) = (1+o(1))\frac{\lambda^\ell}{(\ell-1)!}\exp(-\lambda) = (1+o(1))\frac{(4\varepsilon^2)^\ell}{(\ell-1)!}.$$

For every fixed $\ell \ge 2$, since the random variables $(Z_s^{\ge \ell})_{s \in \mathcal{E}_{a,b}}$ are independent, we may apply Lemma 2.1 with $X_s = Z_s^{\ge \ell}$ and $S = S^{\ge \ell} = \sum_{s \in \mathcal{E}_{a,b}} Z_s^{\ge \ell}$. In other words, $S^{\ge \ell}$ counts the number of vertices in small squares containing at least ℓ vertices. Thus, for every fixed $\ell \ge 2$ we have $\mathbb{E}Z_s^{\ge \ell} = O(\varepsilon^{2\ell}) = o(1)$ and

$$\mathbb{E}S^{\ge \ell} = (1+o(1))|\mathcal{E}_{a,b}|(4\varepsilon^2)^\ell/(\ell-1)! = (1+o(1))\frac{10^{10}r^2(4\varepsilon^2)^{\ell-1}}{(\ell-1)!}. \qquad (6)$$

Since for every $s \in \mathcal{E}_{a,b}$ we have $\mathbb{E}Z_s^{\ge \ell} \le 1$, we may fix $m = 1$. Note also that $Z_s^{\ge \ell}$ attains no value between 0 and 1. Thus, we may simply choose (using the notation introduced right before Lemma 2.1) $\mathcal{L}(Z_s^{\ge \ell}) = \mathcal{L}(Y^{[m]})$ and $T = \sum_{s \in \mathcal{E}_{a,b}} Z_s^{\ge \ell}$; in particular, $\mathbb{E}e^T = \mathbb{E}e^{S^{\ge \ell}}$. Since the random variables $(Z_s^{\ge \ell})_{s \in \mathcal{E}_{a,b}}$ are independent, we have

$$\begin{aligned}\mathbb{E}e^{S^{\ge \ell}} &= \left(\mathbb{E}e^{Z_s^{\ge \ell}}\right)^{|\mathcal{E}_{a,b}|} = \left(\sum_{k=0}^{\ell-1}\lambda^k e^{-\lambda}/k! + \sum_{k \ge \ell}(e\lambda)^k e^{-\lambda}/k!\right)^{|\mathcal{E}_{a,b}|} \le \left(1 + (1+o(1))\frac{(4e\varepsilon^2)^\ell}{\ell!}\right)^{|\mathcal{E}_{a,b}|} \\ &\le \exp\left((1+o(1))\frac{(4e\varepsilon^2)^\ell}{\ell!}|\mathcal{E}_{a,b}|\right).\end{aligned}$$

By Lemma 2.1 applied with $S = S^{\ge \ell}$, we have

$$\begin{aligned}\mathbb{P}\left(S^{\ge\ell} \ge e^{\ell}\mathbb{E}S^{\ge\ell}\right) &\le e^{-e^{\ell}\mathbb{E}S^{\ge\ell}}\mathbb{E}e^{T} = \exp\left(-e^{\ell}(1+o(1))\frac{(4\varepsilon^2)^{\ell}\ell}{\ell!}|\mathcal{E}_{a,b}| + (1+o(1))\frac{(4e\varepsilon^2)^{\ell}}{\ell!}|\mathcal{E}_{a,b}|\right) \\ &= \exp\left(-(1+o(1))\frac{\ell-1}{\ell}\, e^{\ell}\, \mathbb{E}S^{\ge\ell}\right) \le \exp\Big(-(e^2/2+o(1))\mathbb{E}S^{\ge\ell}\Big).\end{aligned}$$

Hence, as long as $\mathbb{E}S^{\ge\ell} \ge \log n$,

$$\mathbb{P}\left(S^{\ge\ell} \ge e^{\ell}\mathbb{E}S^{\ge\ell}\right) = o(1/n^2). \tag{7}$$

Recall that by (6) for every $\ell \ge 2$ we have $\mathbb{E}S^{\ge\ell} = \Theta(r^2\varepsilon^{2\ell-2})$. In particular, $\mathbb{E}S^{\ge 2} = \Theta(r^2\varepsilon^2) = \Omega(r^2\varepsilon_c^2) = \Omega(r^{4/3}) = \Omega(\log^2 n)$. Since $\varepsilon_c \le \varepsilon \le r^{-0.1}$, there exists some integer $\ell_0 \in [3, 11]$ such that $\mathbb{E}S^{\ge j} \ge \log n$ for every integer $j \in [2, \ell_0 - 1]$, whereas $\mathbb{E}S^{\ge \ell_0} < \log n$. Therefore, it follows from (7) that for every integer $j \in [2, \ell_0 - 1]$, with probability $1 - o(1/n^2)$ the value of $S^{\ge j}$ is at most a constant multiplicative factor away from its expectation. Observe that if the number of vertices in one small square is j, where $2 \le j < \ell_0$, then trivially each vertex belongs to exactly $j-1$ pairs of vertices from this square. We get that with probability $1 - o(1/n^2)$ the number of pairs involving such vertices is at most

$$\sum_{j=2}^{\ell_0-1}(j-1)S^{\ge j} = (1+o(1))S^{\ge 2} \le (4e^2\cdot 10^{10} + o(1))r^2\varepsilon^2. \tag{8}$$

On the other hand, one may couple the variables $Z_s^{\ge\ell_0}$ with variables $\hat{Z}_s^{\ge\ell_0}$ in such a way that $Z_s^{\ge\ell_0} \preccurlyeq \hat{Z}_s^{\ge\ell_0}$, and such that $\mathbb{E}\hat{S}^{\ge\ell_0} = \log n$, where $\hat{S}^{\ge\ell_0} := \sum_{s\in\mathcal{E}_{a,b}} \hat{Z}_s^{\ge\ell_0}$. More precisely, we set up the coupling such that for all $k \ge \ell_0$ we have

$$\mathbb{P}\left(\hat{Z}_s^{\ge\ell_0} = k\right) = \frac{\hat{\lambda}^k}{k!}e^{-\hat{\lambda}},$$

where $\hat{\lambda} = 4\hat{\varepsilon}^2$ for some carefully tuned value of $\hat{\varepsilon} \ge \varepsilon$ such that $\mathbb{E}\hat{S}^{\ge\ell_0} = \log n$. (Similarly to the original random variable $Z_s^{\ge\ell_0}$, $\hat{Z}_s^{\ge\ell_0}$ attains no value smaller than ℓ_0 other than 0.) Clearly, $S^{\ge\ell_0} \le \hat{S}^{\ge\ell_0}$ and $\mathbb{E}\, e^{S^{\ge\ell_0}} \le \mathbb{E}e^{\hat{S}^{\ge\ell_0}}$. We may apply Lemma 2.1 again, this time with $T = S = \hat{S}^{\ge\ell_0}$. Arguing as in (7), we get that

$$\mathbb{P}\left(S^{\ge\ell_0} \ge e^{\ell_0}\log n\right) \le \mathbb{P}\left(\hat{S}^{\ge\ell_0} \ge e^{\ell_0}\mathbb{E}\hat{S}^{\ge\ell_0}\right) \le \exp\Big(-(e^2/2+o(1))\mathbb{E}\hat{S}^{\ge\ell_0}\Big) = o(1/n^2).$$

We deduce that with probability $1 - o(1/n^2)$ there are at most $\binom{e^{\ell_0}\log n}{2} \le \binom{e^{11}\log n}{2} \le 10^{10}\log^2 n$ pairs of vertices such that each pair belongs to some small square containing at least ℓ_0 vertices. Combining this observation with (8) we get that with probability $1 - o(1/n^2)$, the number of pairs of vertices that are both contained in one square in the family $\mathcal{E}_{a,b}$ is at most

$$(4e^2\cdot 10^{10} + o(1))r^2\varepsilon^2 + 10^{10}\log^2 n \le 5\cdot 10^{11} r^2\varepsilon^2.$$

(Note that $(4e^2\cdot 10^{10} + o(1))r^2\varepsilon^2 \ge 4320\cdot 10^{10}\log^2 n$ and so the second term is much smaller than the first one.)

Taking a union bound over all four families $\mathcal{E}_{a,b}$, all $O(n)$ squares S, and all $O(\log n)$ values of k, we get that the desired bound holds for the Poisson model with probability $1 - o(1/\sqrt{n})$, and so it holds a.a.s. for $\mathcal{T}(n, r)$.

Parts (c) and (d) are similar to part (b) so we only sketch the proof highlighting a few minor adjustments to the argument. In fact, part (c) follows *exactly* the same argument, since $\varepsilon \le r^{-0.1} = o(1)$ as before. The only thing that is worth pointing out is that the new definition of ε_c, namely, $\varepsilon_c = 12\,\log n/r$ guarantees that $\mathbb{E}S^{\ge 2} = \Theta(r^2\varepsilon^2) = \Omega(r^2\varepsilon_c^2) = \Omega(\log^2 n)$, as needed.

Part (d) requires slightly more careful adjustments since ε_c might *not* tend to zero as $n \to \infty$. As before, the number of vertices in $s \in \mathcal{E}_{a,b}$ is equal to $\mathrm{Po}(\lambda)$, but this time $\lambda = (2\varepsilon_c)^2 = 4\varepsilon_c^2 \le 1/10$ since $r \ge 100 \log n$. We keep the same notation: for $\ell \ge 2$, let $Z_s^{\ge \ell}$ be the random variable counting the number of vertices in s, if this number is at least ℓ, and 0 otherwise. This time we get

$$\mathbb{E}Z_s^{\ge \ell} = \sum_{k \ge \ell} k \cdot \frac{\lambda^k}{k!} \exp(-\lambda) = \frac{C_\ell \cdot \lambda^\ell}{(\ell-1)!} \exp(-\lambda), \text{ where } C_\ell := \sum_{k \ge \ell} \frac{\lambda^{k-\ell}(\ell-1)!}{(k-1)!}.$$

Note that C_ℓ is an explicit constant between 1 and 2 as each term in the sum is at most half of the previous term. It follows that

$$\begin{aligned} \mathbb{E}S^{\ge \ell} &= C_\ell \cdot |\mathcal{E}_{a,b}| \frac{\lambda^\ell}{(\ell-1)!} e^{-\lambda} = (1+o(1)) \frac{10^{10} C_\ell\, r^2 (4\varepsilon_c^2)^{\ell-1}}{(\ell-1)!} e^{-4\varepsilon_c^2} \\ &\le (1+o(1)) \frac{8 \cdot 10^{10} (r\varepsilon_c)^2}{(\ell-1)!} \le \frac{144 \cdot 10^{11} \log^2 n}{(\ell-1)!}. \end{aligned}$$

In particular, $\mathbb{E}S^{\ge 2} = \Theta(\log^2 n)$. Hence, there exists ℓ_0 such that $\frac{j-1}{j} e^j \,\mathbb{E}S^{\ge j} \ge 3 \log n$ for every integer $j \in [2, \ell_0 - 1]$, whereas $\frac{\ell_0 - 1}{\ell_0} e^{\ell_0}\, \mathbb{E}S^{\ge \ell_0} < 3 \log n$. Arguing as before (including the coupling that is needed for the claim for ℓ_0), we get that with probability $1 - o(\ell_0/n^2) = 1 - o(\log n/n^2)$, we have $S^{\ge j} \le e^j \mathbb{E}S^{\ge j}$ for $j < \ell_0$, and $S^{\ge \ell_0} \le e^{\ell_0} \mathbb{E}S^{\ge \ell_0} < \frac{\ell_0}{\ell_0 - 1} \cdot 3 \log n \le 6 \log n$. With probability $1 - o(\log n/n^2)$, the number of pairs of vertices that are both contained in one of the small squares in the family $\mathcal{E}_{a,b}$ is at most

$$\sum_{j=2}^{\ell_0 - 1} (j-1) S^{\ge j} + \binom{S^{\ge \ell_0}}{2} \le \sum_{j=2}^{\ell_0 - 1} \frac{144 \cdot 10^{11} e^j \log^2 n}{(j-2)!} + 18 \log^2 n \le 2 \cdot 10^{15} \log^2 n.$$

The claim may now be deduced after the union bound over the four families $(\mathcal{E}_{a,b})_{a,b \in \{0,1\}}$. □

Now, we are ready to prove Lemma 3.4 and finish the proof of the upper bounds.

Proof of Lemma 3.4. Since we aim for a statement that holds a.a.s., we may assume that the properties stated in Lemma 3.6 and Lemma 3.7 hold. In other words, we do not generate a random graph from $\mathcal{T}(n, r)$ but instead consider a deterministic graph G that satisfies the desired properties. Let $S \in \mathcal{F}$. We will use a non-constructive argument to show that there exists a set $W = X \cup Y \subseteq S$ such that all vertices in $I(S)$ have a unique W-signature.

Let us first concentrate on dense graphs and assume that $r \ge \log^{3/2} n$. We construct a random set X by independently selecting vertices from S to be put into X with probability $\delta = r^{-2/3}$. (Note that this is the only source of randomness at this point as G is a deterministic graph.) By Lemma 3.7(a), the number of vertices in S is at most $2 \cdot 10^{10} r^2$, and so $\mathbb{E}|X| \le 2 \cdot 10^{10}\, r^{4/3}$.

By Lemma 3.7(b), there are at most $2 \cdot 10^{12} r^2 \varepsilon_c^2 = 288 \cdot 10^{12}\, r^{4/3}$ pairs of vertices in $I(S)$ at distance at most ε_c from each other. The number of these is small enough so we do not need to worry about them; all vertices involved in such pairs may simply be put into Y. Fix any $k = k(n) \in \mathbb{N}$ such that $2^k \varepsilon_c \le r^{-0.1} = o(1)$. Concentrate now on any pair of vertices u, v from $I(S)$ that are at distance ε from each other for some $2^{k-1}\varepsilon_c < \varepsilon \le 2^k \varepsilon_c$. By Lemma 3.7(b), there are at most $2 \cdot 10^{12} r^2 (2^k \varepsilon_c)^2 = 288 \cdot 10^{12}\, r^{4/3} \cdot 4^k$ such pairs. By Lemma 3.6, there are at least εr vertices in $\mathcal{B}(u, r) \triangle \mathcal{B}(v, r)$. Since this symmetric difference is included in S, each of these vertices independently ends up in X with probability δ. If at least one of them actually ends up in X, then the vertices u, v are distinguished by the sensors. Hence, u, v are *not* distinguished with probability at most

$$(1-\delta)^{\varepsilon r} \le \exp\left(-\delta \varepsilon r\right) \le \exp\left(-2^{k-1} \delta \varepsilon_c r\right) = \exp\left(-6 \cdot 2^k\right).$$

Similarly, by Lemma 3.7 (a), there are trivially at most $O((r^2)^2) = O(r^4)$ pairs of vertices in $I(S)$ at distance at least $\varepsilon := r^{-0.1}$ from each other. By Lemma 3.6, there are at least $\varepsilon r = r^{0.9}$ vertices in $\mathcal{B}(u,r)\triangle\mathcal{B}(v,r)$ for any such pair of vertices u, v, and so they are *not* distinguished with probability at most

$$(1-\delta)^{r^{0.9}} \le \exp(-\delta r^{0.9}) \le \exp(-r^{0.2}).$$

Combining all of these observations together we get that the expected number of pairs of vertices with the same X-signature is at most

$$\begin{aligned} & 288 \cdot 10^{12}\, r^{4/3} + \sum_{k\ge 1} 288 \cdot 10^{12}\, r^{4/3} \cdot 4^k \cdot \exp\left(-6 \cdot 2^k\right) + O(r^4) \cdot \exp(-r^{0.2}) \\ & \qquad \le 288 \cdot 10^{12}\, r^{4/3} + 0.01 \cdot 288 \cdot 10^{12}\, r^{4/3} + o(1) \le 300 \cdot 10^{12}\, r^{4/3}. \end{aligned}$$

As promised, we put all vertices that occur in at least one such pair into the set Y.

Clearly, by construction each vertex in $I(S)$ has a unique W-signature. Moreover, we get that $\mathbb{E}|W| = \mathbb{E}|X| + \mathbb{E}|Y| \le 2 \cdot 10^{10}\, r^{4/3} + 6 \cdot 10^{14}\, r^{4/3} \le 10^{15}\, r^{4/3}$. Finally, the probabilistic method implies that there exists a set W of size at most $10^{15}\, r^{4/3}$ and the proof for the dense graphs is finished.

Let us now deal with sparser graphs and assume that $100 \log n \le r < \log^{3/2} n$. The proof only requires small adjustments so we only sketch it. We construct a random set X by independently selecting vertices from S to be put into X with probability $\delta = \log^2 n / r^2$ and so, by Lemma 3.7(a), $\mathbb{E}|X| \le 2 \cdot 10^{10} \log^2 n$.

Suppose first that $\log^{5/4} n \le r < \log^{3/2} n$. By Lemma 3.7(c), there are at most $2 \cdot 10^{12} r^2 \varepsilon_c^2 = 288 \cdot 10^{12} \log^2 n$ pairs of vertices in $I(S)$ at distance at most $\varepsilon_c = 12 \log n / r$ from each other. Fix any $k = k(n) \in \mathbb{N}$ such that $2^k \varepsilon_c \le r^{-0.1} = o(1)$, and concentrate on any pair of vertices u, v from $I(S)$ that are at distance ε from each other for some $2^{k-1}\varepsilon_c < \varepsilon \le 2^k \varepsilon_c$. This pair of vertices is *not* distinguished with probability at most

$$(1-\delta)^{\varepsilon r} \le \exp\left(-2^{k-1}\delta\varepsilon_c r\right) = \exp\left(-6 \cdot 2^k \log^3 n / r^2\right) \le \exp\left(-6 \cdot 2^k\right).$$

On the other hand, pairs of vertices that are at distance at least $\varepsilon := r^{-0.1}$ are *not* distinguished with probability at most $(1-\delta)^{r^{0.9}} \le \exp(-\log^2 n / r^{1.1}) \le \exp(-\log^{0.35} n)$. It follows that

$$\begin{aligned} \mathbb{E}|W| &= \mathbb{E}|X| + \mathbb{E}|Y| \le 2 \cdot 10^{10} \log^2 n \\ & \quad + 2\left(288 \cdot 10^{12} \log^2 n + \sum_{k \ge 1} 288 \cdot 10^{12} \log^2 n \cdot 4^k \cdot \exp\left(-6 \cdot 2^k\right) + O(r^4) \cdot \exp(-\log^{0.35} n)\right) \\ &\le 10^{15} \log^2 n. \end{aligned}$$

Suppose then that $100 \log n \le r < \log^{5/4} n$. By Lemma 3.7(d), there are at most $10^{16} \log^2 n$ pairs of vertices in $I(S)$ at distance at most $\varepsilon_c = 12 \log n / r$ from each other. The remaining pairs of vertices are *not* distinguished with probability at most

$$(1-\delta)^{\varepsilon_c r} \le \exp\left(-\delta\varepsilon_c r\right) = \exp\left(-12 \log^3 n / r^2\right) \le \exp\left(-12 \log^{1/2} n\right).$$

This time

$$\begin{aligned} \mathbb{E}|W| &\le (2 \cdot 10^{10} \log^2 n) + 2\left(10^{16} \log^2 n + O(r^4) \cdot \exp(-12 \log^{1/2} n)\right) \\ &\le 3 \cdot 10^{16} \log^2 n. \end{aligned}$$

The upper bound for very sparse graphs is trivial. If $r < 100 \log n$, then one may simply put sensors on all vertices in S, that is, take $\delta = 1$. The bound follows immediately from Lemma 3.7(a). $\square$

4 Lower bound

This section is devoted to the proof of the lower bounds stated in Theorem 1.1. Assume first that $r = r(n) \geq \log n$; we will adjust the argument to sparser graphs at the end of this section. Let $\mathcal{B}_R$ be the ball with radius $r/3$, centered in the center O of the square $[0, \sqrt{n}]^2$. We will show that if the number of sensors is less than the lower bound given by Theorem 4.1, a.a.s. the robber has a strategy to remain undetected forever while staying in the ball $\mathcal{B}_R$ during the entire game.

Theorem 4.1. *Fix $r = r(n) \geq \log n$ and let $G \in \mathcal{T}(n, r)$. Then, a.a.s. the robber may remain undetected forever in $\mathcal{B}_R$ in the presence of less than $10^{-4} r^{4/3}/\log^{1/3} n$ sensors at each round.*

The general idea behind the proof of the lower bound is quite natural and intuitive. First, for a carefully tuned function $\varepsilon = \varepsilon(r)$, we will show that there are relatively many pairs of vertices in $\mathcal{B}_R$ that are at distance at most ε from each other. In fact, in order to simplify the argument we will concentrate on a particular *special* sub-family of pairs of such vertices, which we will call *special pairs*, that satisfy some additional useful property. On the other hand, we will show that regardless of where a single sensor is placed, it distinguishes only a few pairs of such vertices. This will immediately imply the desired lower bound for the number of sensors needed to distinguish vertices in $\mathcal{B}_R$ and so to locate the robber hiding in $\mathcal{B}_R$.

A family of pairs of vertices from $\mathcal{B}_R$ that are at distance at most ε from each other is called *ε-special* if each vertex in $\mathcal{B}_R$ belongs to at most one such pair. In other words, an ε-special family induces a matching. We will start by showing that there exists a large ε-special family, provided that the graph is dense enough.

Lemma 4.2. *Fix $r = r(n) \geq \log n$ and let $G \in \mathcal{T}(n, r)$. Fix $\varepsilon = \varepsilon(n) = (\log n/r)^{1/3} \leq 1$. Then, a.a.s. there exists an ε-special family of pairs of vertices of size $r^2\varepsilon^2/100$.*

Proof. It will be convenient to use the de-Poissonization technique as explained in Section 2.3. We start with tessellating the entire torus into squares of side length $\varepsilon/\sqrt{2}$. Trivially, any two vertices that belong to the same square are at distance at most ε. Since the area of each square (namely, $\varepsilon^2/2 \leq 1/2$) is negligible in comparison to the area of the ball $\mathcal{B}_R$ (namely, $\pi r^2/9 \geq \pi \log^2 n/9$), the number of squares that are completely inside the ball is equal to $\ell = (2\pi/9 + o(1))(r^2/\varepsilon^2)$.

We construct a special family of pairs as follows. We independently expose the vertices in each square that is completely inside the ball, and if exactly two vertices belong to a given square, then we add this pair to the family. The probability that a given square has exactly two vertices in it is equal to

$$p := \frac{(\varepsilon^2/2)^2}{2!} \exp(-\varepsilon^2/2) \geq \frac{\varepsilon^4}{8e^{1/2}}.$$

Hence, the number of special pairs in $\mathcal{B}_R$ is stochastically bounded from below by the random variable $X \sim \mathrm{Bin}(\ell, p)$ with $\mathbb{E}X = \ell p \geq r^2\varepsilon^2/36 = \Theta((\log n)^{2/3} r^{4/3}) = \Omega(\log^2 n)$. It follows immediately from Chernoff's bound (2) that with probability $1 - o(1/\sqrt{n})$ the size of our ε-special family is at least $\mathbb{E}X/2 \geq r^2\varepsilon^2/100$. By de-Poissonization the same property holds a.a.s. in $\mathcal{T}(n, r)$ and so the proof of the lemma is finished. $\square$

Suppose that a sensor is placed on a vertex $v \in V$ of a connected geometric graph. For a given non-negative integer i, let $D_i(v)$ be the set of vertices that are at graph distance i from v in G. Since vertices in $\mathcal{B}_R$ induce a complete graph, putting a sensor on v divides $\mathcal{B}_R \cap V$ into the set of vertices in $\mathcal{B}_R \cap D_k(v)$ (possibly empty) at distance k from v and the set of vertices in $\mathcal{B}_R \cap D_{k+1}(v)$ (again, possibly empty) that are at distance $k+1$ from v, where $k \in \mathbb{N} \cup \{0\}$: indeed, if a vertex u in $\mathcal{B}_R$ is at graph distance k from v, then every other vertex in $\mathcal{B}_R$ is at graph distance at most $k+1$ from v. Note that, in particular, if $v \in \mathcal{B}_R$, then $D_0(v) = \{v\}$ and $D_1(v)$ contains all other vertices in $\mathcal{B}_R$, so this sensor only distinguishes itself from the remaining vertices in $\mathcal{B}_R$. The partition of $\mathcal{B}_R \cap V$ is more challenging to investigate when the sensor is placed on a vertex $v \in V \setminus \mathcal{B}_R$ so that $k \neq 0$. Let us concentrate on this situation.

Note that all vertices in $D_k(v)$ belong to

$$U(D_{k-1}(v)) := \bigcup_{u \in D_{k-1}(v)} \mathcal{B}(u, r).$$

The argument that will be used in the proof of Theorem 4.1 will show that $U(D_{k-1}(v))$ has a non-empty intersection with $\mathcal{B}_R$. On the other hand, no vertex in $D_{k+1}(v)$ belongs to $U(D_{k-1}(v))$. Hence, in order to estimate the number of ε-special pairs of vertices that are distinguished by v we need to concentrate on the *boundary* between the sets $\mathcal{B}_R \cap U(D_{k-1}(v))$ and $\mathcal{B}_R \setminus U(D_{k-1}(v))$. Note also that every vertex in an ε-special pair that is distinguished by v must be at Euclidean distance at most ε from the boundary of $\mathcal{B}_R \cap U(D_{k-1}(v))$ in $\mathcal{B}_R$, a key property that will be used to estimate the number of such pairs.

In order to investigate the "shape" of the boundary, we relax the assumption about $D_{k-1}(v)$ (vertices at distance $k-1$ from v) and simply assume that it is *any* finite set of points $\{O_i\}_{i \in \mathcal{I}}$ with the property that $U(\{O_i\}_{i \in \mathcal{I}})$ has nonempty intersection with $\mathcal{B}_R$. Note that there could be many disconnected boundary regions of $\mathcal{B}_R \setminus U(\{O_i\}_{i \in \mathcal{I}})$, and it will be convenient to distinguish between small and large regions. For the latter family of regions, we could use Weyl's famous tube formula (see Chapter 17.2 in [14]), but we do not do that for the following two reasons. First of all, we decided to provide an independent proof to keep the presentation self-contained. Moreover, the application of Weyl's formula would require introducing certain curvature concepts from differential geometry, which would make the argument comparable in length to our elementary proof but slightly more technical.

Let us now start with a few preparatory geometric lemmas.

Lemma 4.3. *Let k_1 and k_2 be two circles with centers O_1 and O_2 and radii $r_1 \leq r_2$, respectively. Suppose that $k_1 \cap k_2 = \{A, B\}$ and let ℓ_1, ℓ_2 be two lines, parallel to O_1O_2, that divide the plane in three parts such that A, O_1 and B are all in different parts. Let $\ell_1 \cap k_1 = \{P_1, S_1\}$, $\ell_2 \cap k_1 = \{Q_1, R_1\}$, $\ell_1 \cap k_2 = \{P_2, S_2\}$ and $\ell_2 \cap k_2 = \{Q_2, R_2\}$ such that P_1, Q_1 are on the same side of the line AB and also P_2, Q_2 are on the same side of AB. Then, $|P_1Q_1| \geq |P_2Q_2|$.*

Proof. Without loss of generality, we may assume that either O_1 and O_2 are on different sides of the line AB, or $O_1 \in AB$ (otherwise, apply symmetry to k_2 with respect to AB - this will not change the lengths of both P_1Q_1 and P_2Q_2), and we may also assume that $d(O_1, \ell_1) = h_1$ and $d(O_1, \ell_2) = h_2$ with $h_1 \geq h_2$, see Figure 3. We have that

$$\begin{aligned}
|P_1Q_1| &= \frac{h_1 + h_2}{\sin(\angle P_1Q_1R_1)} = \frac{h_1 + h_2}{\sin(\angle P_1O_1R_1/2)} \\
&= \frac{h_1 + h_2}{\sin(\pi/2 - \angle O_1P_1S_1/2 + \angle O_1Q_1R_1/2)} \\
&= \frac{h_1 + h_2}{\cos(\angle O_1P_1S_1/2 - \angle O_1Q_1R_1/2)},
\end{aligned}$$

and also

$$\begin{aligned}
|P_2Q_2| &= \frac{h_1 + h_2}{\sin(\angle P_2Q_2R_2)} = \frac{h_1 + h_2}{\sin(\angle P_2O_2R_2/2)} \\
&= \frac{h_1 + h_2}{\sin(\pi/2 - \angle O_2P_2S_2/2 + \angle O_2Q_2R_2/2)} \\
&= \frac{h_1 + h_2}{\cos(\angle O_2P_2S_2/2 - \angle O_2Q_2R_2/2)}.
\end{aligned}$$

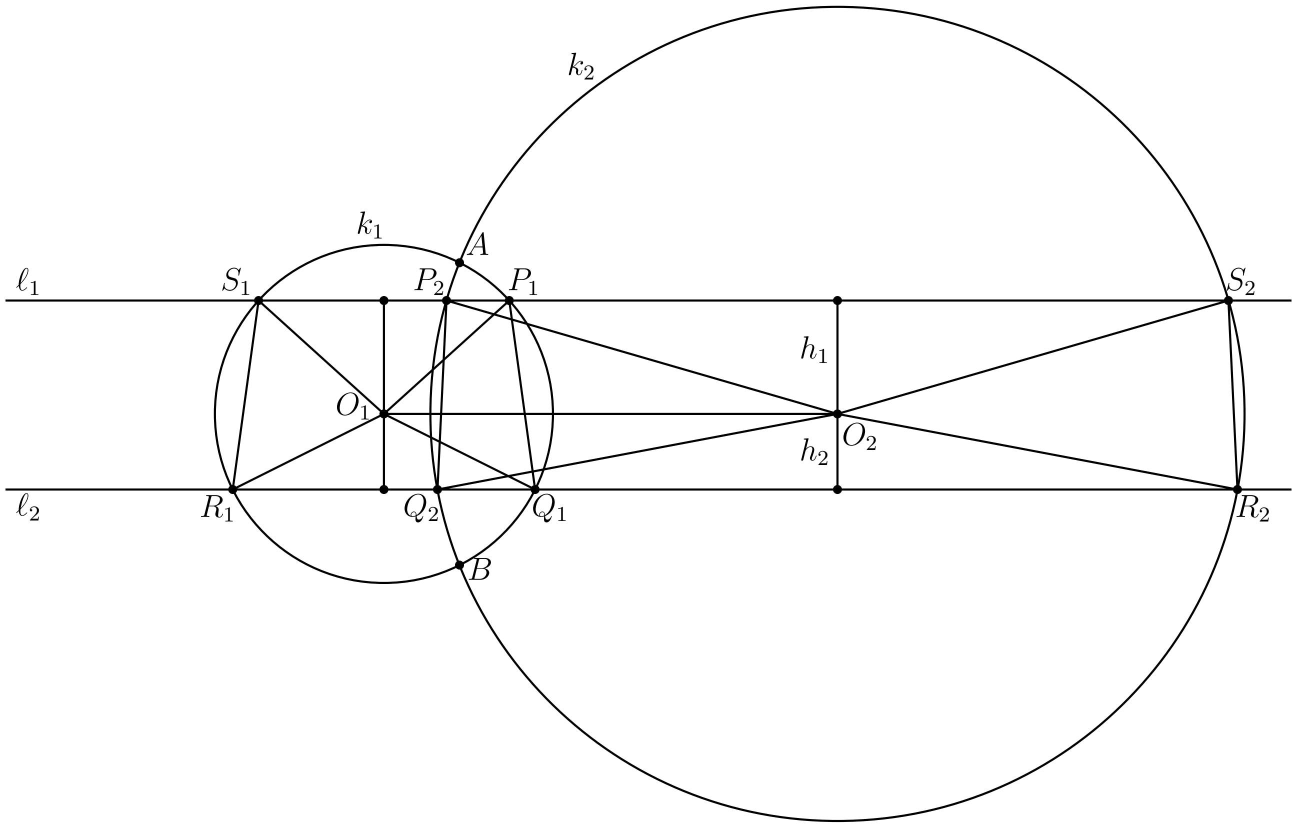

Figure 3: Illustration of the proof of Lemma 4.3.

Note that since $h_1 \geq h_2$ we have that

$$\angle O_1P_1S_1 = \arcsin(h_1/r_1) \geq \arcsin(h_2/r_1) = \angle O_1Q_1R_1$$

and

$$\angle O_2P_2S_2 = \arcsin(h_1/r_2) \geq \arcsin(h_2/r_2) = \angle O_2Q_2R_2.$$

Moreover, standard analysis shows that the function $f : r \in [h_1, +\infty) \mapsto \arcsin(h_1/r) - \arcsin(h_2/r)$ is decreasing and therefore

$$\angle O_2P_2S_2 - \angle O_2Q_2R_2 = f(r_2) \leq f(r_1) = \angle O_1P_1S_1 - \angle O_1Q_1R_1.$$

We conclude that

$$|P_2Q_2| = \frac{h_1 + h_2}{\cos(f(r_2)/2)} \leq \frac{h_1 + h_2}{\cos(f(r_1)/2)} = |P_1Q_1|,$$

and the proof of the lemma is completed. □

Corollary 4.4. *In the setting of Lemma 4.3, the length of the arc* $\overset{\frown}{Q_1P_1}$ *in* k_1 *is larger than or equal to the length of the arc* $\overset{\frown}{P_2Q_2}$ *in* k_2*.*

Proof. This follows immediately from the fact that by Lemma 4.3 $|Q_1P_1| \geq |P_2Q_2|$, and that the curvature of the cycle k_1 is larger than or equal to the curvature of the cycle k_2 (recall that $r_1 \leq r_2$). □

The above corollary can be used to bound the length of the boundary. The next lemma is the key observation that we will use for this purpose.

Lemma 4.5. *Fix a finite set of points $\{O_i\}_{i\in\mathcal{I}}$ and a ball $\mathcal{B}'=\mathcal{B}(O',r')$, where $r'\le r/3$. Assume that the points $O'\cup\{O_i\}_{i\in\mathcal{I}}$ are in general position. Then, the length of the boundary between the set*

$$\mathcal{R}_1 := \mathcal{B}'\cap\left(\bigcup_{i\in\mathcal{I}}\mathcal{B}(O_i,r)\right)$$

and its complement $\mathcal{R}_2:=\mathcal{B}'\setminus\mathcal{R}_1$ is at most the length of the perimeter of $\mathcal{B}'$, that is, is at most $2\pi r'$.

Proof. For convenience, let $\partial\mathcal{R}$ denote the boundary of region $\mathcal{R}$ and let $\mathcal{B}_i=\mathcal{B}(O_i,r)$ for every $i\in\mathcal{I}$. Let us consider the following transformation of $\partial\mathcal{R}_1$. For any $i\in\mathcal{I}$ and any arc in $\partial\mathcal{R}_1\cap\partial\mathcal{B}_i$, carry this arc along two rays starting at its endpoints, parallel to the ray $\overrightarrow{O'O_i}$, and in the same direction as this ray, to an arc of $\partial\mathcal{B}'$. For example, there are three such arcs in the top part of Figure 4, drawn in red; for example, the arc $\overset{\frown}{A_1X}\subseteq\partial\mathcal{B}_1$ is projected alongside $\overrightarrow{O'O_1}$ to the arc $\overset{\frown}{X_1A_1}\subseteq\partial\mathcal{B}'$ (note that we use the convention that arcs are always taken in anticlockwise direction).

First, let us note that by Corollary 4.4 the image of every arc in $\partial\mathcal{R}_1$ is at least as long as the arc itself. Hence, it remains to prove that the images of different arcs are disjoint. If $|\mathcal{I}|=1$, then the statement trivially holds and so we may assume that $|\mathcal{I}|\ge 2$. Note also that it is sufficient to prove this fact for any pair of arcs in $\partial\mathcal{R}_1$ that belong to two different balls from the family $\{\mathcal{B}_i\}_{i\in\mathcal{I}}$, say $\mathcal{B}_1$ and $\mathcal{B}_2$. Recall that $\mathcal{B}_1$ has center O_1 and suppose that it intersects $\mathcal{B}'$ in A_1 and B_1. Similarly, recall that $\mathcal{B}_2$ has center O_2 and suppose that it intersects $\mathcal{B}'$ in A_2 and B_2. See the bottom part of Figure 4 for illustration.

Suppose that $\overset{\frown}{B_1A_1}\subseteq\partial\mathcal{B}'$ and $\overset{\frown}{B_2A_2}\subseteq\partial\mathcal{B}'$ have a non-empty intersection in $\mathcal{B}'$; otherwise, the statement clearly holds. Then, the arcs $\overset{\frown}{A_1B_1}\subseteq\partial\mathcal{B}_1$ and $\overset{\frown}{A_2B_2}\subseteq\partial\mathcal{B}_2$ intersect in a unique point $X\in\mathcal{B}_R$. Let $X_1\in\partial\mathcal{B}'$ be the point such that $XX_1||O'O_1$ with $\overrightarrow{XX_1}$ having the same direction as $\overrightarrow{O'O_1}$. Similarly, let $X_2\in\partial\mathcal{B}'$ be the point such that and $\overrightarrow{XX_2}||\overrightarrow{O'O_2}$ with $\overrightarrow{XX_2}$ having the same direction as $\overrightarrow{O'O_2}$ (see, again, the bottom part of Figure 4). Then, $\partial\mathcal{R}_1\cap\partial\mathcal{B}_1$ is either contained in the arc $\overset{\frown}{A_1X}$ or in the arc $\overset{\frown}{XB_1}$ of $\partial\mathcal{B}_1$. We may assume that $\partial\mathcal{R}_1\cap\partial\mathcal{B}_1$ is contained in the arc $\overset{\frown}{A_1X}$ of $\partial\mathcal{B}_1$, as the other case can be dealt with analogously. Then, $\partial\mathcal{R}_1\cap\partial\mathcal{B}_2$ is contained in the arc $\overset{\frown}{XB_2}$ of $\partial\mathcal{B}_2$. In the rest of the proof, all arcs belong to $\partial\mathcal{B}'$. Our goal is to prove that the arcs $\overset{\frown}{X_1A_1}$ and $\overset{\frown}{B_2X_2}$ are disjoint. To show this we perform a continuous rotation of $\mathcal{B}_2$ around the point X in the direction which decreases the length of the (directed) arc $\overset{\frown}{B_2B_1}$, until $\mathcal{B}_2$ coincides with $\mathcal{B}_1$. This operation decreases the length of the arc $\overset{\frown}{X_2X_1}$ as well. More importantly, at the end of this rotation when $\mathcal{B}_2$ coincides with $\mathcal{B}_1$, the arc $\overset{\frown}{B_1X_1}$ becomes the image of the arc $\overset{\frown}{B_2X_2}$. This proves that the arcs $\overset{\frown}{X_1A_1}$ and $\overset{\frown}{B_2X_2}$ of $\mathcal{B}'$ were initially disjoint. This finishes the proof of the lemma since $|\partial\mathcal{B}'|=2\pi r'$. □

We will also need the following fact that has been known for centuries and by now has become part of the mathematical folklore.

Lemma 4.6 (Folklore; see for example [15])**.** *Out of all connected open sets in the plane with a given perimeter, the circle has the largest area. In other words, each connected open set of perimeter ℓ has area at most $\ell^2/4\pi$.*

Recall that $\{O_i\}_{i\in\mathcal{I}}$ is assumed to be a finite set of points that partitions $\mathcal{B}_R$ into

$$\mathcal{R}_1 := \mathcal{B}_R\cap\left(\bigcup_{i\in\mathcal{I}}\mathcal{B}(O_i,r)\right) \qquad\text{and}\qquad \mathcal{R}_2:=\mathcal{B}_R\setminus\mathcal{R}_1.$$

For example, in the bottom part of Figure 4, the region $\mathcal{R}_1$ is bounded by the arcs $\overset{\frown}{A_1X}$, $\overset{\frown}{XB_2}$ and $\overset{\frown}{B_2A_1}$ from the circles with centers O_1, O_2 and O', respectively. (Eventually, $\{O_i\}_{i\in\mathcal{I}}$ will be fixed to be $D_{k-1}(v)$, the set of vertices that are at distance $k-1$ from v for some $k\in\mathbb{N}$.) Note that $\mathcal{R}_2$ does not need to be

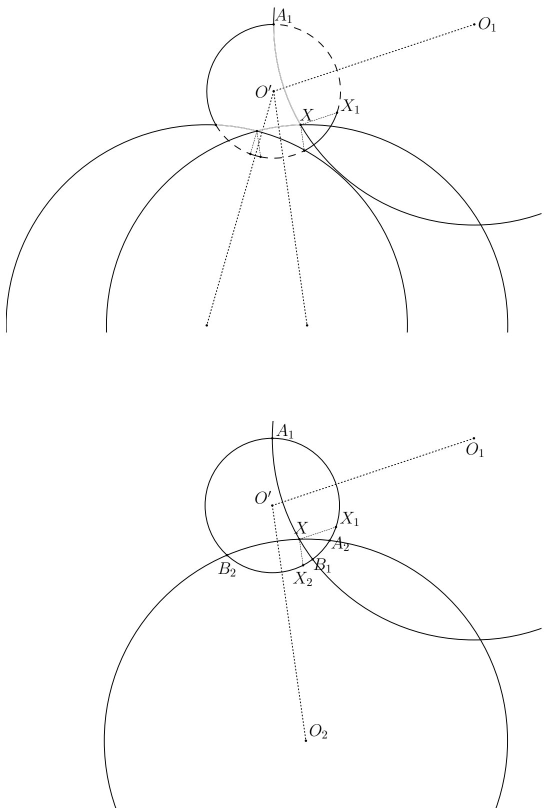

Figure 4: The top figure shows how the internal grey arcs are carried over to the external dashed arcs. The bottom figure shows an illustration of the proof that the external arcs are disjoint.

a connected set. However, since the number of balls $(\mathcal{B}(O_i, r))_{i\in\mathcal{I}}$ is finite, the number of contact points between their boundaries is also finite, and therefore the number of connected components of $\mathcal{R}_2$ must also be finite. A component of $\mathcal{R}_2$ will be called *large* if its boundary has length more than $6\pi\varepsilon$, and it will be called *small* otherwise.

Let us first concentrate on small components and consider the union of them. Suppose that for some $k \in \mathbb{N}$ there are k small components with lengths of their boundaries $\ell_1, \ell_2, \dots, \ell_k$. By Lemma 4.5, we get that

$$\sum_{i=1}^{k} \ell_i \leq \frac{2\pi r}{3},$$

and thus using Lemma 4.6 we deduce that the area of the union of all small regions is at most

$$\sum_{i=1}^{k} \frac{\ell_i^2}{4\pi} \leq \frac{3\varepsilon}{2} \sum_{i=1}^{k} \ell_i \leq \pi r \varepsilon. \tag{9}$$

Now we may concentrate on large components. Let $\gamma \subseteq \mathbb{R}^2$ be a closed curve. The *ε-tube* $t_\varepsilon(\gamma)$ around γ is the set of points $Q \in \mathbb{R}^2$ such that $d_E(\gamma, Q) \leq \varepsilon$. Moreover, for any arc $\widehat{a}$ of a circle c with radius at least $r/3 > 2\varepsilon$, define the *ε-cut tube* $t^c_\varepsilon(\widehat{a})$ around $\widehat{a}$ as the intersection of $t_\varepsilon(c)$ with the sector of c, corresponding to the arc $\widehat{a}$. In the next observation, the diameter of an arc is the longest (Euclidean) distance between some two points in this arc.

Observation 4.7. *Let A, B be two points inside a ball $\mathcal{B}$ with radius r_1. Let $\widehat{AB}$ be an arc between A and B with diameter $|AB|$, which is part of a circle c with radius $r_2 > r_1$. Then, $\widehat{AB} \subseteq \mathcal{B}$.*

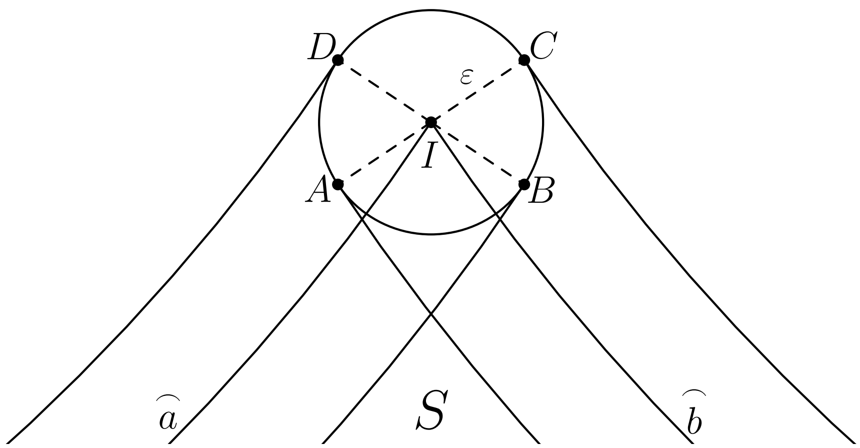

Figure 5: Illustration of the proof of Lemma 4.8. Here, I is an intersection point of the arcs $\widehat{a}$ and $\widehat{b}$ on the boundary of the large component S. The points O_1 and O_2 are not shown since they are too far and S is contained between $\widehat{a}$ and $\widehat{b}$.

Proof. Since the radius of c is larger than the radius of $\mathcal{B}$ and $A, B \in \mathcal{B}$, c and $\partial\mathcal{B}$ must intersect in two points. Thus, either the arc $\widehat{AB}$ is entirely contained in $\mathcal{B}$ or it starts in $\mathcal{B}$, contains every point in $c \cap \mathcal{B}^c$ and then ends in $\mathcal{B}$. In the second case, the diameter of the arc $\widehat{AB}$ would be $2r_2 > 2r_1$, which would lead to a contradiction since $|AB|$ is clearly at most $2r_1$. Thus, $\widehat{AB} \subseteq \mathcal{B}$. □

Lemma 4.8. *For every large component S of $\mathcal{R}_2$ we have that the area of $t_\varepsilon(\partial S)$ is at most the sum of the areas of the ε-cut tubes around the arcs participating in ∂S, that is, at most $2\varepsilon|\partial S|$.*

Proof. The claim is trivial if $\mathcal{R}_2 = \mathcal{B}_R$. Otherwise, consider an arbitrary intersection point I of two circles c_1 and c_2 with centers O_1 and O_2, respectively. Let $\widehat{a}$ and $\widehat{b}$ be the two arcs in ∂S, contained in the circles c_1 and c_2, respectively, which contain I as an endpoint. Let A, I, C, O_1 be collinear points, lying on the line IO_1 in this order and such that $|AI| = |IC| = \varepsilon$. Let also B, I, D, O_2 be collinear points, lying on the line IO_2 in this order and such that $|BI| = |ID| = \varepsilon$. Then, define the *internal sector at I*, denoted by $IS_S(I, \varepsilon)$, to be the sector AIB of the ball $\mathcal{B}(I, \varepsilon)$, and also define the *external sector at I*, denoted by $ES_S(I, \varepsilon)$, to be the sector CID of the ball $\mathcal{B}(I, \varepsilon)$. Then, the tube $t_\varepsilon(\partial S)$ is obtained as a union of all cut tubes of the arcs in ∂S and the external sectors at all intersection points of neighbouring arcs - see Figure 5.

We need the following two claims before proceeding with the proof of the lemma.

Claim 4.9. *There exist two internal sectors without common points.*

Proof of the claim. Suppose for a contradiction that each pair of internal sectors intersect. Fix one internal sector with center I. Then, by the triangle inequality every intersection point of two neighbouring arcs in ∂S is at distance at most 2ε from I. Since $r/3 > 2\varepsilon$, by Observation 4.7 every arc in ∂S is contained in $\mathcal{B}(I, 2\varepsilon)$ and therefore $S \subseteq \mathcal{B}(I, 2\varepsilon)$. If $\partial\mathcal{B}_R \cap \partial S = \emptyset$, Lemma 4.5 with $\mathcal{B}' = \mathcal{B}(I, 2\varepsilon)$ implies that the perimeter of S is at most $|\partial\mathcal{B}(I, 2\varepsilon)| = 4\pi\varepsilon$, contradicting with the fact that S is a large component. Otherwise, the perimeter of S is bounded from above by the sum of $|\partial S \cap \partial\mathcal{B}_R|$ and the perimeter of the region containing S, in $\mathcal{B}(I, 2\varepsilon) \cap \left(\bigcup_{i\in\mathcal{I}} \mathcal{B}_i\right)^c$. Roughly speaking, the region described above is obtained by "taking out" the ball $\mathcal{B}_R$. First, note that $\partial S \cap \partial\mathcal{B}_R \subseteq \mathcal{B}(I, 2\varepsilon) \cap \partial\mathcal{B}_R$ and therefore, since the curvature of $\partial\mathcal{B}_R$ is smaller than the curvature of $\partial\mathcal{B}(I, 2\varepsilon)$, we have $|\partial S \cap \partial\mathcal{B}_R| \le |\mathcal{B}(I, 2\varepsilon) \cap \partial\mathcal{B}_R| \le |\partial\mathcal{B}(I, 2\varepsilon)|/2 =$

$2\pi\varepsilon$. Second, one may apply Lemma 4.5 to $\mathcal{B}(I, 2\varepsilon) \cap \left(\bigcup_{i\in\mathcal{I}} \mathcal{B}_i\right)^c$ with $\mathcal{B}' = \mathcal{B}(I, 2\varepsilon)$ to conclude that the perimeter of $\mathcal{B}(I, 2\varepsilon) \cap \left(\bigcup_{i\in\mathcal{I}} \mathcal{B}_i\right)^c$ is at most $|\partial\mathcal{B}(I, 2\varepsilon)| = 4\pi\varepsilon$. In total we get $|\partial S| \le 6\pi\varepsilon$, which again is a contradiction with the fact that S is a large component. □

Claim 4.10. *Let $\ell \in \mathbb{N}$. If a point P is contained in ℓ internal sectors, it must be contained in the ε-cut tubes around at least $\ell + 1$ of the arcs in ∂S.*

Proof of the claim. By Claim 4.9, there exists an internal sector $IS_S(I, \varepsilon)$, which does not contain P. Let us enumerate the arcs along ∂S, starting from one of the arcs, incident to I, and finish with the other arc, incident to I. Let $I_1, I_2, \ldots, I_\ell$ be the points, for which $P \in \cap_{i\in[\ell]} IS_S(I_i, \varepsilon)$. Then, P is contained in the ε-cut tube around every arc of ∂S, incident to any of $I_1, I_2, \ldots, I_\ell$, and there are at least $\ell + 1$ such arcs. □

By Claim 4.10, one may directly deduce that

$$\sum_{\widehat{a}\in\partial S} |t^c_\varepsilon(\widehat{a})| \ge \left|\bigcup_{\widehat{a}\in\partial S} t^c_\varepsilon(\widehat{a})\right| + \sum_{I:\, I=\widehat{a}\cap\widehat{b};\, \widehat{a},\widehat{b}\in\partial S} |IS_S(I, \varepsilon)|$$

$$= \left|\bigcup_{\widehat{a}\in\partial S} t^c_\varepsilon(\widehat{a})\right| + \sum_{I:\, I=\widehat{a}\cap\widehat{b};\, \widehat{a},\widehat{b}\in\partial S} |ES_S(I, \varepsilon)| \ge |t_\varepsilon(S)|.$$

The proof of the lemma is finished. □

Corollary 4.11. *The union of all ε-tubes around the boundaries of large components has area at most $4\pi\varepsilon r/3$.*

Proof. This follows from Lemma 4.8, applied for every large component of $\mathcal{R}_2$, and Lemma 4.5, which states that the union of the boundaries of these components has length at most $2\pi r/3$. □

Let us now come back to the proof of the lower bound for the localization number of dense graphs.

Proof of Theorem 4.1. Fix $r = r(n) \ge \log n$ and $\varepsilon = \varepsilon(n) = (\log n/r)^{1/3} \le 1$. Let $\mathcal{B}_R$ be the ball of radius $r/3$ centered in the center O of the square $[0, \sqrt{n}]^2$. It follows from Lemma 4.2 that a.a.s. there exists an ε-special family of pairs of vertices of size $r^2\varepsilon^2/100$ in $\mathcal{B}_R$. We will show that a.a.s. a single sensor placed on a vertex v cannot distinguish more than $100\varepsilon^3 r$ special pairs which will imply the desired lower bound of $(r^2\varepsilon^2/100)/(100\varepsilon^3 r) = 10^{-4}\, r/\varepsilon = 10^{-4}\, r^{4/3}/\log^{1/3} n$.

Indeed, suppose that the cops must use less than $10^{-4}\, r^{4/3}/(\log n)^{1/3}$ sensors in each round. Using the notation from Section 2.1, when the cops start the game by putting their sensors on S_1, at least two vertices (namely, some special pair) in $\mathcal{B}_R$ have the same S_1-signature. The robber may choose the equivalence class $R^1_{j_1}$ these two vertices belong to and remain undetected in the very first round. Suppose now that $R^{i-1}_{j_{i-1}}$ contains at least two vertices from $\mathcal{B}_R$. In round i, once the cops choose S_i, we get the partition $N[R^{i-1}_{j_{i-1}}] = R^i_1 \cup R^i_2 \cup \ldots \cup R^i_{\ell_i}$ with every vertex in R^i_j having the same S_i-signature. Since the ball $\mathcal{B}_R$ has radius $r/3$, $N[R^{i-1}_{j_{i-1}}]$ includes *all* vertices in $\mathcal{B}_R$. Hence, again, the robber may choose some $R^i_{j_i}$ of size at least 2 as there is at least one special pair of vertices in $\mathcal{B}_R$ with the same S_i-signature. It follows that $|R^i_{j_i}| \ge 2$ for all i and so the robber has a winning strategy.

It remains to show that a.a.s. a single sensor placed on a vertex v cannot distinguish more than $100\varepsilon^3 r$ special pairs. Clearly, if v is in $\mathcal{B}_R$, then it can distinguish at most one special pair, the one including the vertex v itself. Hence, we may concentrate on sensors placed on vertices outside of the ball the robber is hiding at. To that end, we will use de-Poissonization technique as explained in Section 2.3 and show that the desired property holds with probability $1 - o(n^{-2})$ for a given vertex v outside of $\mathcal{B}_R$. The desired conclusion will hold by a union bound over all vertices.

Let v be any vertex of $G \in \mathcal{T}(n, r)$ that is outside of the ball $\mathcal{B}_R$. We will carefully expose the graph in a *breadth-first-search* fashion. Recall that $D_i(v)$ denotes the set of vertices at graph distance i from v. We start with $D_0(v) = \{v\}$. Iteratively, as long as no vertex in $D_i(v)$ is at Euclidean distance at most r from $\mathcal{B}_R$ we do the following. Since vertices in $D_{i+1}(v)$ must belong to

$$U(D_i(v)) := \bigcup_{u \in D_i(v)} \mathcal{B}(u, r),$$

we expose all vertices in the part of $U(D_i(v))$ that is not exposed yet. Vertices that are found there form the set $D_{i+1}(v)$. We stop the process prematurely if no vertex is found in $U(D_i(v))$; in this case v does not distinguish *any* special pair and so the desired property holds. Suppose that for some $k \in \mathbb{N}$ we stopped the process because for the first time some vertex $w \in D_{k-1}(v)$ is at Euclidean distance at most r from $\mathcal{B}_R$. If w is in fact at distance at most $r/3$ from $\mathcal{B}_R$, then all points in $\mathcal{B}_R$ are at distance at most r from w. It follows that, despite the fact that we did not expose the ball yet, we can safely claim that all vertices in $\mathcal{B}_R$ will end up in $D_k(v)$. On the other hand, if no vertex in $D_{k-1}(v)$ is at Euclidean distance at most $r/3$ from $\mathcal{B}_R$, then we may pick the point A that is on the segment between w and the center O of the ball $\mathcal{B}_R$ and at distance, say, $r/2$ from O. By Observation 2.4, we may assume that there is a vertex at distance at most $2\sqrt{\log n} = o(r)$ from A. (This is a standard technique in the theory of random graphs but it is quite delicate. We wish to use the properties guaranteed a.a.s. by Observation 2.4, but we also wish to avoid working in a conditional probability space, as doing so would make the necessary probabilistic computations intractable. Thus, we will work in the unconditional probability space but in our argument we assume that the properties mentioned in the observation hold. Since these properties hold a.a.s., the probability of the set of outcomes in which our argument does *not* apply to is $o(1)$, and thus can be safely excised at the end of the argument.) This vertex is not only adjacent to w but also all points in $\mathcal{B}_R$ are at distance at most r from it. Hence, this time we can safely claim that all vertices in $\mathcal{B}_R$ will end up in $D_k(v) \cup D_{k+1}(v)$.

Let us summarize the current situation: $D_{k-1}(v)$ is a set of vertices at distance $k-1$ from v that partitions $\mathcal{B}_R$ into

$$\mathcal{R}_1 := \mathcal{B}_R \cap \left(\bigcup_{w \in D_{k-1}(v)} \mathcal{B}(w, r) \right) \qquad \text{and} \qquad \mathcal{R}_2 := \mathcal{B}_R \setminus \mathcal{R}_1.$$

The region $\mathcal{R}_1$ is non-empty but $\mathcal{R}_2$ might be empty. The ball $\mathcal{B}_R$ is not exposed yet but we do know that all vertices in $\mathcal{R}_1$ (if there are any) will end up in $D_k(v)$ and all vertices in $\mathcal{R}_2$ (again, if there are any) will end up in $D_{k+1}(v)$. In order for a pair of vertices (a, b) to be distinguished by v, one of the two vertices (say, a) has to be in $\mathcal{R}_1$ and the other one (say, b) has to be in $\mathcal{R}_2$. More importantly, if a and b are at distance at most ε from each other, b has to belong to some small component of $\mathcal{R}_2$ or to some ε-tube around the boundary of some large component (but still in $\mathcal{R}_2$).

Let us expose vertices in $\mathcal{R}_2$. By (9) and Corollary 4.11, we get that the total number of vertices in all small components and in the union of all ε-tubes around the boundaries of large components is stochastically bounded from above by the random variable $X \sim \text{Po}(\lambda)$ with $\lambda := \pi\varepsilon r + 4\pi\varepsilon r/3 = 7\pi\varepsilon r/3$. We have

$$\begin{aligned} \mathbb{P}\Big(X \geq 2\lambda\Big) &= \sum_{i \geq 2\lambda} \frac{\lambda^i}{i!} e^{-\lambda} \leq 2\,\mathbb{P}\Big(X = 2\lambda\Big) = 2\,\frac{\lambda^{2\lambda}}{(2\lambda)!} e^{-\lambda} \\ &\leq 2\,\frac{\lambda^{2\lambda}}{(2\lambda/e)^{2\lambda}} e^{-\lambda} = 2\,\left(\frac{e}{4}\right)^{\lambda} \leq 2\,\exp\left(-\frac{\lambda}{3}\right). \end{aligned} \tag{10}$$

Since $r \geq \log n$, we get that $\lambda = \frac{7\pi}{3}\varepsilon r = \frac{7\pi}{3} r^{4/3}/\log^{1/3} n \geq \frac{7\pi}{3}\log n > 7\log n$. It follows from (10) that with probability $1 - o(n^{-2})$, $X \leq 2\lambda$. Each vertex b that appears in this region eliminates at most one special pair but this itself is not enough to get the desired bound.

We condition on the event that there are at most $2\lambda = 14\pi\varepsilon r/3$ vertices b in $\mathcal{R}_2$ that can potentially participate in ε-special pairs and argue as follows. Since the associated vertices a have to be not only in $\mathcal{R}_1$ but also at distance at most ε from some vertex b in $\mathcal{R}_2$, we expose vertices in $\mathcal{R}_1$ and check how many of them are close to some vertex b in $\mathcal{R}_2$. The number of such vertices a is stochastically bounded from above by the random variable $Y \sim \mathrm{Po}(\xi)$ with $\xi := (14\pi\varepsilon r/3)(\pi\varepsilon^2) = 14\pi^2\varepsilon^3 r/3 = (14\pi^2/3)\log n > 46\log n$. It follows from (10) that with probability $1-o(n^{-2})$, $Y \leq 2\xi = 28\pi^2\varepsilon^3 r/3 < 100\varepsilon^3 r$. Each such vertex a that appears eliminates at most one special pair, and so the desired bound holds, and the proof is finished. $\square$

Note that the previous proof gives the lower bounds of Part 1 and Part 2 in Theorem 1.1. Let us now consider sparser graphs. We first adjust the argument used above that gives a matching lower bound for r very close to $\log n$ (Part 3) in Theorem 1.1), namely, we assume first that

$$\frac{\log n}{(\log\log n)^{1/2}\log\log\log n} \leq r \leq \log n.$$

By fixing $\varepsilon = 1$, we argue as in the proof of Lemma 4.2 that a.a.s. there exists a 1-special family of pairs of vertices of size $r^2/100$. The argument then proceeds as before: the total number of vertices in $\mathcal{R}_2$ in all small components and in the union of all 1-tubes around the boundaries of large components is stochastically bounded from above by the random variable $X \sim \mathrm{Po}(\lambda)$ with $\lambda := 7\pi r/3$. This time we fix

$$\beta := \frac{200\log n}{\log(e\log n/r)} > 2\lambda$$

and notice that

$$\begin{aligned}
\mathbb{P}\Big(X \geq \beta\Big) \leq 2\,\mathbb{P}\Big(X = \beta\Big) = 2\,\frac{\lambda^\beta}{\beta!}e^{-\lambda} \leq \frac{\lambda^\beta}{(\beta/e)^\beta} = \left(\frac{7e\pi r}{3\beta}\right)^\beta = \exp\left(-\beta\log\left(\frac{3\beta}{7e\pi r}\right)\right) \\
= \exp\left(-\frac{200\log n}{\log(e\log n/r)}\log\left(\frac{600\log n/\log(e\log n/r)}{7e\pi r}\right)\right) \\
\leq \exp\left(-\frac{200\log n}{\log(e\log n/r)}\log\left(\frac{e\log n/r}{\log(e\log n/r)}\right)\right) \\
= \exp\left(-200\log n\left(1 - \frac{\log\log(e\log n/r)}{\log(e\log n/r)}\right)\right) = o(1/n^2).
\end{aligned}$$

Arguing as before, we get the lower bound of $(r^2/100)/\beta = \Theta(r^2\log(e\log n/r)/\log n)$. Note that since $\varepsilon = 1$ (so $r\varepsilon = r\varepsilon^3$), we have that the order of the number of points in $\mathcal{R}_2$ that X counts would be comparable to the number of 1-special pairs that one sensor may distinguish, and therefore the last stage in the proof of Part 1 and Part 2 of Theorem 1.1 will not contribute and may be omitted.

Let us now concentrate on even sparser graphs for which we use a different argument. Tessellate the torus $\mathcal{T}_n$ into a family of squares $(S_i)_{i\in\mathcal{I}}$ of side lengths $3r$ and with centers $(O_i)_{i\in\mathcal{I}}$. Then, for every $i \in \mathcal{I}$, let $\mathcal{B}_i$ be the ball of radius $\log n/16r$ with center O_i. Finally, for every $i \in \mathcal{I}$, let $\mathcal{R}_i$ be the set of points P, for which $\mathcal{C}(P,r)\cap\mathcal{B}_i \neq \emptyset$. The idea of this part is the following: for every square S_i of the tessellation and the ball $\mathcal{B}_i$ inside S_i, note that the region $\mathcal{R}_i$ consists of the points that may distinguish some vertices inside $\mathcal{B}_i$. We will show that for many squares S_i, the corresponding region $\mathcal{R}_i$ contains no vertex of G, and at the same time there is one such square S_i with many vertices inside $\mathcal{B}_i$. Since the robber can jump from one vertex to another inside the ball, she can only be trapped by putting a sensor on each vertex inside the ball. We fill in the details and start with the following preliminary result:

Lemma 4.12. *For every $i \in \mathcal{I}$, $\mathcal{R}_i$ has area $\pi\log n/4$ and is disjoint from $\mathcal{B}_i$.*

Proof. For every $i \in \mathcal{I}$, $\mathcal{R}_i$ consists of all points at distance between $r - \log n/16r$ and $r + \log n/16r$ from O_i, so $|\mathcal{R}_i| = \pi(r+\log n/16r)^2 - \pi(r - \log n/16r)^2 = \pi \log n/4$. Moreover, $2 \cdot \log n/16r < r$ since $r \geq \sqrt{\log n}$, so $\mathcal{R}_i \cap \mathcal{B}_i = \emptyset$. □

Proof of the lower bound of Theorem 1.1, Part 4). Expose the region $\bigcup_{i\in\mathcal{I}} \mathcal{R}_i \subseteq \bigcup_{i\in\mathcal{I}} S_i \setminus \mathcal{B}_i$. Note that the regions $(\mathcal{R}_i)_{i\in\mathcal{I}}$ are disjoint and for any $i \in \mathcal{I}$ the probability that no vertex of $G \in \mathcal{T}(n,r)$ falls into $\mathcal{R}_i$ is by Lemma 4.12, $\exp(-\pi \log n/4) = 1/n^{\pi/4}$. Let $\mathcal{J}$ be the family of indices i, for which $\mathcal{R}_i$ does not contain any vertices of G. We conclude that the family of variables $(\mathbb{1}_{\mathcal{R}_i \cap V(G)=\emptyset})_{i\in\mathcal{I}}$ consists of independent Bernoulli variables with parameter $1/n^{\pi/4}$, so by Chernoff's bound $|\mathcal{J}| \geq n^{1-\pi/4}/18r^2$ with probability $1 - o(1/\sqrt{n})$. We condition on this event.

Now, what remains is to give a lower bound that holds with probability $1-o(1/\sqrt{n})$ for the maximum of the $|\mathcal{J}| \geq n^{1-\pi/4}/18r^2 \geq n^{0.2}$ (the last inequality holds for every large enough n) Poisson variables with mean $\lambda = \pi \log^2 n/(16r)^2$, representing the number of vertices of G in the balls $(\mathcal{B}_j)_{j\in\mathcal{J}}$. Set $\xi := \dfrac{\log n}{50 \log(r^2/\log n)}$. By a direct computation we get for every $r \geq r_0$ that $\xi \geq 2\lambda$ and that

$$\begin{aligned}
\mathbb{P}\left(\forall j \in \mathcal{J}, |B_j| \leq \xi\right) &= \prod_{j\in\mathcal{J}} \mathbb{P}\left(|B_j| \leq \xi\right) \\
&\leq \prod_{j\in\mathcal{J}} 2\mathbb{P}\left(|B_j| = \xi\right) \\
&= \prod_{j\in\mathcal{J}} 2\exp\left(-\lambda\right) \frac{\lambda^\xi}{\xi!} \\
&\leq 2^{n^{0.2}} \exp\left(-n^{0.2}\lambda\right) \left(\frac{\lambda^\xi}{\xi!}\right)^{n^{0.2}} \\
&\leq 2^{n^{0.2}} \exp\left(-n^{0.2}\lambda\right) \left(\frac{e\lambda}{\xi}\right)^{n^{0.2}\xi} \\
&= 2^{n^{0.2}} \exp\left(-n^{0.2}\frac{\pi\log^2 n}{(16r)^2}\right) \left(\frac{e\pi \log^2 n/(16r)^2}{\log n/(50\log(r^2/\log n))}\right)^{\frac{n^{0.2}\log n}{50\log(r^2/\log n)}} \\
&= 2^{n^{0.2}} \exp\left(-n^{0.2}\frac{\pi\log^2 n}{256r^2}\right) \left(\frac{50e\pi\log(r^2/\log n)}{256r^2/\log n}\right)^{\frac{n^{0.2}\log n}{50\log(r^2/\log n)}} \\
&= \left(2\exp\left(\frac{\log n}{50\log(r^2/\log n)}\log\left(\frac{50e\pi\log(r^2/\log n)}{256r^2/\log n}\right) - \frac{\pi\log^2 n}{256r^2}\right)\right)^{n^{0.2}}.
\end{aligned}$$

We conclude that the last probability is $o(1/\sqrt{n})$ since

$$\frac{\log n}{50\log(r^2/\log n)}\log\left(\frac{50e\pi\log(r^2/\log n)}{256r^2/\log n}\right) < 0 \text{ and } \frac{\pi\log^2 n}{256r^2} \geq 1$$

for every $r = o(\log n)$ and $r \geq r_0$, which gives an upper bound of $(2/e)^{n^{0.2}} = o(1/\sqrt{n})$.

By de-Poissonization we deduce that a.a.s. there is a ball $\mathcal{B}_i$ among $(\mathcal{B}_j)_{j\in\mathcal{J}}$ containing at least ξ vertices of the random geometric graph $G \in \mathcal{T}(n,r)$. In particular, since by definition of the set $\mathcal{J}$ the vertices $\mathcal{B}_i \cap V(G)$ cannot be distinguished by a sensor outside $\mathcal{B}_i$, the robber can always escape from the cops in the presence of only $\xi - 2 = \Omega(\log n/\log(r^2/\log n))$ sensors by choosing to remain in the ball $\mathcal{B}_i$ after each step, thereby finishing the proof of the lower bound of Part 4) in Theorem 1.1. □

5 Outlook and open problems

In this paper we determined up to a multiplicative poly-logarithmic factor the localization number of the random geometric graph. As already mentioned in the introduction, for any graph G, we have $\zeta(G) \leq \beta(G)$. Whereas in $\mathcal{G}(n,p)$ these two parameters are relatively close to each other, for many values of r this is not the case for $G \in \mathcal{T}(n,r)$, as the following lemma shows. In fact, in view of the lower bound on $\zeta(G)$ given by Part 1) in Theorem 1.1, the following lemma shows that for $r \ll n^{3/10}$ the bounds are far from each other. For the sake of completeness, we also show the upper bound our approach for the localization number gives for $\beta(G)$.

Lemma 5.1. *Let $G \in \mathcal{T}(n,r)$. A.a.s. we have*

(i) If $r \gg 1$ and $r \leq c\sqrt{n/\log n}$ for small enough $c > 0$, then $\beta(G) = \Omega(n/r^2)$.

(ii) If $\log^{3/2} n \leq r \leq \sqrt{n}/4$, then $\beta(G) = O((n \log^{2/3} n)/r^{2/3})$.

Proof. We prove part (i) in a Poissonized setup, de-Poissonizing only at the end. Tessellate the torus into square cells of width $3r$, and consider in each cell C the inner cell c of width $0.1r$ centered at the same point as C. Subdivide further c into subcells of width $1/r$. Consider the event $\mathcal{E}_C$ that inside the inner cell c of C there is a subcell having exactly two vertices u, v, and that there is no vertex at distance at most r from u (v, respectively), while at the same time being at distance more than r from v (u, respectively). Observe that if $\mathcal{E}_C$ holds, then either u or v has to be taken into a minimum set of sensors which guarantees that the cops can win in one round. Moreover, for different cells C, C' the corresponding events $\mathcal{E}_C, \mathcal{E}_{C'}$ are independent. Denote by X_C the indicator random variable for $\mathcal{E}_C$.

The probability that no subcell has exactly 2 vertices therein is equal to

$$\left(1 - \frac{(1/r^2)^2}{2} e^{-1/r^2}\right)^{(0.01+o(1))r^4} = \left(1 - \frac{0.5 + o(1)}{r^4}\right)^{(0.01+o(1))r^4} = e^{-0.005} + o(1).$$

Condition on the event that there is a subcell having exactly two vertices u, v, and observe that $\mathcal{B}(u,r)\Delta\mathcal{B}(v,r)$ is completely contained in $C \setminus c$. Thus, since by Observation 3.5, $|\mathcal{B}(u,r)\Delta\mathcal{B}(v,r)| \leq 4 \cdot \sqrt{2}(1 + o(1))$, we have

$$\mathbb{P}(X_C = 1) \geq (1 - e^{-0.005} + o(1)) \cdot (e^{-4\sqrt{2}} + o(1)) \geq 10^{-5}.$$

Denote $X = \sum_C X_C$. Observing that there are $\Theta(n/r^2)$ cells implies that $\mathbb{E}(X) = \Theta(n/r^2)$. Hence, since $r \leq c\sqrt{n/\log n}$ for a small enough constant $c > 0$, part (i) follows by Chernoff's bound (2) together with the de-Poissonization argument given in Section 2.3.

For part (ii), recall by Lemma 3.6 that a.a.s. for any pair of points with positions A, B such that $d_T(A,B) \geq \varepsilon := (\log n/r)^{1/3}$, the number of vertices in $\mathcal{B}(A,r)\Delta\mathcal{B}(B,r)$ is at least $\min(\varepsilon, 2r)r$. Now, for every vertex, put a sensor on it independently of all others, with probability $C \log^{2/3} n/r^{2/3}$ for some large enough constant $C > 3$ (thus constructing a random set of expected size $Cn \log^{2/3} n/r^{2/3}$), and then add for any pair of vertices that is not distinguished yet one of the two vertices. We now show that the number of vertices added is at most of the same order, thus proving the desired upper bound of $O(n \log^{2/3} n/r^{2/3})$. To do so observe that by considering the family $\mathcal{F}$ of squares (defined right before the statement of Lemma 3.4) every pair of vertices at distance at most $0.1r$ is inside one square $S \in \mathcal{F}$. Hence, for $r \geq \log^{3/2} n$, by Lemma 3.7 (b) (note that $(\log n/r)^{1/3} \leq r^{-0.1}$ for the range of r) together with a union bound over all squares S in $\mathcal{F}$, the number of pairs at distance at most $(\log n/r)^{1/3}$ is at most $|\mathcal{F}| \cdot (2 + o(1)) \cdot 10^7 r^{4/3} \log^{2/3} n = O(n \log^{2/3} n/r^{2/3})$, and we may add for each such pair one vertex. For all other pairs of vertices at distance at least $(\log n/r)^{1/3}$, by a union bound, the probability that there exists a pair not distinguished by the random set is at most

$$n^2 \left(1 - \frac{C \log^{2/3} n}{r^{2/3}}\right)^{r^{2/3} \log^{1/3} n} = o(1/n),$$

and hence a.a.s. all such vertices will be distinguished by the random set. $\square$

Together with the already obtained lower bounds on $\zeta(G)$ and using the fact that $\zeta(G) \le \beta(G)$, we thus have the following bounds on the metric dimension:

Theorem 5.2. *Let $G \in \mathcal{T}(n,r)$. A.a.s. the following bounds hold:*

- *If $1 \ll r \le \sqrt{n}/4$, then $\Omega\left(\max(n/r^2, r^{4/3}/\log^{1/3} n)\right) = \beta(G)$.*
- *If $\log^{3/2} n \le r \le \sqrt{n}/4$, then $\beta(G) = O\left(n \log^{2/3} n / r^{2/3}\right)$.*

We finish the paper with the following natural open questions.

Open problem 5.3. *Let $G \in \mathcal{G}(n,r)$. Theorem 5.2 implies that a.a.s. $\beta(G) = n^{2/3+o(1)}$, provided that $r = n^{1/2+o(1)}$ (and $r \le \sqrt{n}/4$) but the bounds are far away from each other for sparser graphs. What is the value of $\beta(G)$ for $G \in \mathcal{G}(n,r)$?*

Open problem 5.4. *Let $G \in \mathcal{G}(n,r)$. Our results imply that a.a.s. $\zeta(G)/\beta(G) = o(1)$, provided that $r \ll n^{3/10}$. What about denser graphs?*

Open problem 5.5. *Let $G \in \mathcal{G}(n,r)$. Our results give relatively tight bounds for the localization number of G, provided that $r \ge \log^{3/2} n$. The bounds for sparser graphs are slightly worse. For example, our lower bound in the range of $r \in [r_0, \log n]$ is not monotonic but there is no apparent reason why it should not be monotonic. Moreover, what is the localization number close to the threshold of connectivity?*

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