

SIZE-RAMSEY NUMBERS OF CYCLES VERSUS A PATH

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ABSTRACT. The size-Ramsey number $\hat{R}(\mathcal{F}, H)$ of a family of graphs \mathcal{F} and a graph H is the smallest integer m such that there exists a graph G on m edges with the property that any coloring of the edges of G with two colors, say, red and blue, yields a red copy of a graph from \mathcal{F} or a blue copy of H . In this paper we first focus on $\mathcal{F} = \mathcal{C}_{\leq cn}$, where $\mathcal{C}_{\leq cn}$ is the family of cycles of length at most cn , and $H = P_n$. In particular, we show that $2.00365n \leq \hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq 31n$. Using similar techniques, we also managed to analyze $\hat{R}(C_n, P_n)$, which was investigated before but until last year only by using the regularity method.

1. INTRODUCTION

Following standard notations, for any family of graphs \mathcal{F} and any graph H , we write $G \rightarrow (\mathcal{F}, H)$ if any coloring of the edges of G with 2 colors, red and blue, yields a red copy of some graph from \mathcal{F} or a blue copy of H . For simplicity, we write $G \rightarrow (F, H)$ if $\mathcal{F} = \{F\}$ and $G \rightarrow F$ instead of $G \rightarrow (F, F)$. We define the *size-Ramsey number* of the pair (\mathcal{F}, H) as

$$\hat{R}(\mathcal{F}, H) = \min\{|E(G)| : G \rightarrow (\mathcal{F}, H)\}$$

and again, for simplicity, $\hat{R}(F, H) = \hat{R}(\{F\}, H)$ and $\hat{R}(F) = \hat{R}(F, F)$.

One of the most studied directions in this area is the size-Ramsey number of P_n , a path on n vertices. It is obvious that $\hat{R}(P_n) = \Omega(n)$ and that $\hat{R}(P_n) = O(n^2)$ (for example, $K_{2n} \rightarrow P_n$), but the exact behaviour of $\hat{R}(P_n)$ was not known for a long time. In fact, Erdős [10] offered \$100 for a proof or disproof that

$$\hat{R}(P_n)/n \rightarrow \infty \quad \text{and} \quad \hat{R}(P_n)/n^2 \rightarrow 0.$$

This problem was solved by Beck [1] in 1983 who, quite surprisingly, showed that $\hat{R}(P_n) < 900n$. (Each time we refer to inequality such as this one, we mean that the inequality holds for sufficiently large n .) A variant of his proof, provided by Bollobás [7], gives $\hat{R}(P_n) < 720n$. Very recently, different and more elementary arguments were used by the first and the third author of this paper [8, 9], and by Letzter [14] that show that $\hat{R}(P_n) < 137n$ [8], $\hat{R}(P_n) < 91n$ [14], and $\hat{R}(P_n) < 74n$ [9], respectively. On the other hand, the first nontrivial lower bound was provided by Beck [2] and his result was subsequently improved by Bollobás [6] who showed that $\hat{R}(P_n) \geq (1 + \sqrt{2})n - O(1)$; today we know that $\hat{R}(P_n) \geq 5n/2 - O(1)$ [9].

For any $c \in \mathbb{R}_+$, let $\mathcal{C}_{\leq cn}$ be the family of cycles of length at most cn . In this paper, we continue to use similar ideas as in [8, 14, 9] to deal with $\hat{R}(\mathcal{C}_{\leq cn}, P_n)$. Such techniques (very simple but quite powerful) were used for the first time in [3, 4]; see also recent book [15] that

The first author was supported in part by Simons Foundation Grant #522400.

The third author was supported in part by NSERC.

covers several tools including this one. Corresponding theorems use different approaches and different probability spaces that might be interesting on their own rights. Some non-trivial lower bounds are provided as well. In particular, it is shown that

$$2.00365n \leq \hat{R}(\mathcal{C}_{\leq n}, P_n) \leq 31n$$

for sufficiently large n .

We also study $\hat{R}(C_n, P_n)$ and show that for even and sufficiently large n we have

$$5n/2 - O(1) \leq \hat{R}(C_n, P_n) \leq 2257n.$$

In fact, the lower bound for odd values of n can be improved to $3(n-1)$. The linearity of $\hat{R}(C_n, P_n)$ also follows from the earlier result of Haxell, Kohayakawa and Łuczak [11] who proved that the size-Ramsey number $\hat{R}(C_n, C_n)$ is linear in n . However, their proof is based on the regularity method and therefore the leading constant is enormous. Very recently, Javadi, Khoeini, Omidi, and Pokrovskiy [13] showed that $\hat{R}(C_n, C_n) < 10^{12}n$ avoiding a use of the regularity lemma.

2. PRELIMINARIES

Let us recall a few classic models of random graphs that we study in this paper. The *binomial random graph* $\mathcal{G}(n, p)$ is the random graph G with vertex set $[n] := \{1, 2, \dots, n\}$ in which every pair $\{i, j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p . The *binomial random bipartite graph* $\mathcal{G}(n, n, p)$ is the random bipartite graph $G = (V_1 \cup V_2, E)$ with partite sets V_1, V_2 , each of order n , in which every pair $\{i, j\} \in V_1 \times V_2$ appears independently as an edge in G with probability p . Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

Recall that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Since we aim for results that hold a.a.s., we will always assume that n is large enough.

Another probability space that we are interested in is the probability space of random d -regular graphs with uniform probability distribution. This space is denoted by $\mathcal{G}_{n,d}$, and asymptotics are for $n \rightarrow \infty$ with $d \geq 2$ fixed, and n even if d is odd.

Instead of working directly in the uniform probability space of random regular graphs on n vertices $\mathcal{G}_{n,d}$, we use the *pairing model* (also known as the *configuration model*) of random regular graphs, first introduced by Bollobás [5], which is described next. Suppose that dn is even, as in the case of random regular graphs, and consider dn points partitioned into n labelled buckets v_1, v_2, \dots, v_n of d points each. A *pairing* of these points is a perfect matching into $dn/2$ pairs. Given a pairing P , we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the buckets v_1, v_2, \dots, v_n , and a pair $\{x, y\}$ in P corresponds to an edge $v_i v_j$ in $G(P)$ if x and y are contained in the buckets v_i and v_j , respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{-(d^2-1)/4}$ depending on d , so that any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$. For this reason, asymptotic

results over random pairings suffice for our purposes. For more information on this model, see, for example, the survey of Wormald [17].

Also, we will be using the following well-known concentration inequality. Let $X \in \text{Bin}(n, p)$ be a random variable with the binomial distribution with parameters n and p . Then, a consequence of Chernoff's bound (see, for example, Theorem 2.1 in [12]) is that for any $t \geq 0$

$$\mathbb{P}(X \leq \mathbb{E}X - t) \leq \exp\left(-\mathbb{E}X \varphi\left(\frac{-t}{\mathbb{E}X}\right)\right) \leq \exp\left(-\frac{t^2}{2\mathbb{E}X}\right),$$

where $\varphi(x) = (1+x)\log(1+x) - x$ for $x \geq -1$ (and $\varphi(x) = \infty$ for $x < -1$).

For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible to the asymptotic calculations we will make. Finally, we use $\log n$ to denote natural logarithms.

3. UPPER BOUNDS

In this section, we will use the following observation.

Lemma 3.1. *Let $c \in \mathbb{R}_+$ and let G be a graph of order $(c+1)n$. Suppose that for every two disjoint sets of vertices S and T such that $|S| = |T| = cn/2$ we have $e(S, T) \geq cn$. Then, $G \rightarrow (\mathcal{C}_{\leq cn}, P_n)$.*

The above lemma will quickly follow from the following result that was first noticed by Ben-Eliezer, Krivelevich and Sudakov [3, 4] and later by Pokrovskiy [16], Dudek and Prałat [8, 9], and Letzter [14].

Lemma 3.2. *For every graph G there exist two disjoint subsets $S, T \subseteq V(G)$ of equal size such that there are no edges between them and $G \setminus (S \cup T)$ has a Hamilton path.*

Proof of Lemma 3.1. Consider any red-blue coloring of the edges of any graph G of order $(c+1)n$. Assume that there is no blue copy of P_n . Now by Lemma 3.2 (applied to the blue subgraph of G) we obtain that there are sets S and T such that $|S| = |T| = (|V(G)| - n)/2 = cn/2$ and there are no blue edges between S and T . However, by assumption $e(S, T) \geq cn$. So $G[S, T]$ (the bipartite graph induced by two partite sets S and T), which consists only of red edges, is not a forest and so must contain a red cycle of length at most cn . \square

Since random graphs are good expanders, after carefully selecting parameters, they should allow the desired graphs and so they should provide some upper bounds for the corresponding size Ramsey numbers. Let us start with investigating binomial random graphs. The next result is fairly standard. We write $A_n \approx B_n$ to mean that $A_n = (1 + o(1))B_n$ as $n \rightarrow \infty$.

Lemma 3.3. *Let $c \in \mathbb{R}_+$ and let $d = d(c) > 4/c$ be such that*

$$(c+1)\log(c+1) + c\log(d/2) - c^2d/4 + c = 0.$$

Then, the following two properties hold a.a.s. for $G \in \mathcal{G}((c+1)n, d/n)$:

- (i) *for every two disjoint sets of vertices S and T such that $|S| = |T| = cn/2$ we have $e(S, T) > cn$;*
- (ii) *$|E(G)| \approx d(c+1)^2n/2$.*

Proof. Let S and T with $|S| = |T| = cn/2$ be fixed and let $X = X_{S,T} = e(S, T)$, that means, X is the random variable that counts the number of edges between sets S and T . Clearly, $\mathbb{E}X = c^2dn/4 > cn$ and by Chernoff's bound

$$\begin{aligned} \Pr(X \leq cn) &\leq \exp\left(-\mathbb{E}X \left(\frac{cn}{\mathbb{E}X} \log\left(\frac{cn}{\mathbb{E}X}\right) + \frac{\mathbb{E}X - cn}{\mathbb{E}X}\right)\right) \\ &= \exp\left((c \log(cd/4) - c^2d/4 + c)n\right). \end{aligned}$$

Thus, by the union bound over all choices of S and T we have

$$\begin{aligned} \Pr\left(\bigcup_{S,T} (X_{S,T} \leq cn)\right) &\leq \binom{(c+1)n}{cn/2} \binom{(c+1)n - cn/2}{cn/2} \exp\left((c \log(cd/4) - c^2d/4 + c)n\right) \\ &= \frac{((c+1)n)!}{(cn/2)!(cn/2)!n!} \exp\left((c \log(cd/4) - c^2d/4 + c)n\right). \end{aligned}$$

Using Stirling's formula ($x! \approx \sqrt{2\pi x}(x/e)^x$) we get

$$\begin{aligned} \Pr\left(\bigcup_{S,T} (X_{S,T} \leq cn)\right) &= o\left(\exp\left(\left((c+1) \log(c+1) - c \log(c/2) + c \log(cd/4) - c^2d/4 + c\right)n\right)\right) \\ &= o\left(\exp\left(\left((c+1) \log(c+1) + c \log(d/2) - c^2d/4 + c\right)n\right)\right) = o(1), \end{aligned}$$

by the definition of d . This implies part (i).

Part (ii) follows immediately from Chernoff's bound as the expected number of edges in $\mathcal{G}((c+1)n, d/n)$ is $\binom{(c+1)n}{2}d/n \approx d(c+1)^2n/2$. \square

Getting numerical upper bounds for size Ramsey numbers is a straightforward implication of the previous two lemmas. We get the following result.

Theorem 3.4. *Let $c \in \mathbb{R}_+$. Then, for all sufficiently large n we have*

$$\hat{R}(\mathcal{C}_{\leq cn}, P_n) < \begin{cases} \frac{80 \log(e/c)}{c} n & \text{for } 0 < c < 1 \\ 37n & \text{for } c \geq 1. \end{cases}$$

Proof. Let $c \in \mathbb{R}_+$ and let $d = d(c) > 4/c$ be such that

$$(c+1) \log(c+1) + c \log(d/2) - c^2d/4 + c = 0. \quad (1)$$

Then, an immediate consequence of Lemma 3.1 and Lemma 3.3 is that a.a.s. $G \in \mathcal{G}((c+1)n, d/n) \rightarrow (\mathcal{C}_{\leq cn}, P_n)$. As a result, for any $\varepsilon > 0$ and sufficiently large n ,

$$\hat{R}(\mathcal{C}_{\leq cn}, P_n) < \left(\frac{d(c+1)^2}{2} + \varepsilon\right)n. \quad (2)$$

In particular, for sufficiently large n , it follows that $\hat{R}(\mathcal{C}_{\leq n}, P_n) < 37n$ and so by monotonicity $\hat{R}(\mathcal{C}_{\leq cn}, P_n) < 37n$ for any $c \geq 1$.

In order to get an explicit upper bound in the case $0 < c < 1$, we will show that $d = d(c) < \hat{d} := 40 \log(e/c)/c$. From this the result will follow since $d(c+1)^2/2 \leq 2d <$

$2\hat{d} = 80 \log(e/c)/c$. (Observe that this bound is of the right order as it is easy to see that $d = \Omega(\log(e/c)/c)$ as $c \rightarrow 0$.)

Note that

$$\begin{aligned} f(c, \hat{d}) &= (c + 1) \log(c + 1) + c \log(\hat{d}/2) - c^2 \hat{d}/4 + c \\ &\leq (c + 1)c + c \log(20/c^2) - 10c \log(e/c) + c, \end{aligned}$$

as for $c \in (0, 1]$ we have $\log(c + 1) \leq c$ and $\log(e/c) \leq 1/c$ (and so $\hat{d} \leq 40/c^2$). Now, since clearly $\log(e/c) \geq 1$,

$$\begin{aligned} f(c, \hat{d}) &\leq 3c \log(e/c) + c \log(20/c^2) - 10c \log(e/c) \\ &< 3c \log(e/c) + c \log(e^3/c^3) - 10c \log(e/c) = -4c \log(e/c) < 0. \end{aligned}$$

It follows that $d < \hat{d}$, and the proof is complete. □

It follows from this proof that the best upper bound on $\hat{R}(\mathcal{C}_{\leq cn}, P_n)$ is given by (2), where $d = d(c)$ is defined in (1). Unfortunately, d has no explicit form as a function of c . Let $\alpha = \alpha(c) = d(c + 1)^2/2$. Thus, for sufficiently large n , we have $\hat{R}(\mathcal{C}_{\leq cn}, P_n) < \lceil \alpha(c) \rceil n$. On Figure 1 we present $\alpha(c)$ computed for several values of c .

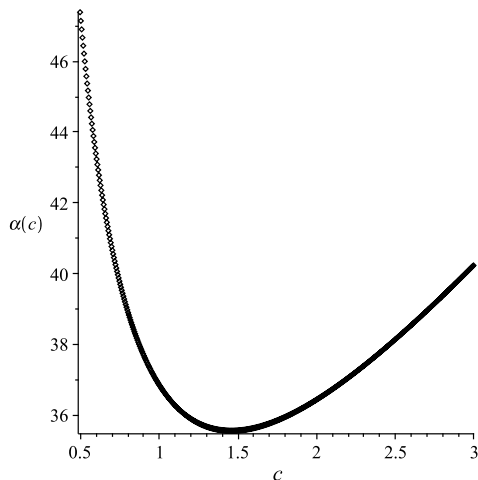


FIGURE 1. Graph of $\alpha(c)$ as a function of c .

Now, let us investigate random d -regular graphs. For simplicity we focus on the $c = 1$ case. Similarly like in [8, 9] this model yields slightly better upper bound for the size Ramsey numbers.

First we show a similar result to Lemma 3.3.

Lemma 3.5. *Let $d = 31$. Then, a.a.s. for every $G \in \mathcal{G}_{2n,d}$ and every two disjoint sets of vertices S and T of G with $|S| = |T| = n/2$ we have $e(S, T) > n$.*

Proof. Consider $G \in \mathcal{G}_{2n,d}$. Our goal is to show that the expected number of pairs of two disjoint sets, S and T , such that $|S| = |T| = n/2$ and $e(S, T) < n$ tends to zero as $n \rightarrow \infty$.

Let $a = a(n)$ be any function of n such that $an \in \mathbb{Z}$ and $0 \leq a \leq 1$ and $b = b(n)$ be any function of n such that $bn \in \mathbb{Z}$ and $0 \leq b \leq d/2 - a$. Let $X(a, b)$ be the expected

number of pairs of two disjoint sets S, T such that $|S| = |T| = n/2$, $e(S, T) = an$, and $e(S, V \setminus (S \cup T)) = bn$. Using the pairing model, it is clear that

$$X(a, b) = \binom{2n}{n/2} \binom{3n/2}{n/2} \binom{dn/2}{an}^2 (an)! \binom{dn/2 - an}{bn} \binom{dn}{bn} (bn)! \\ \cdot M(dn/2 - an - bn) \cdot M((dn/2 - an) + (dn - bn)) / M(2dn),$$

where $M(i)$ is the number of perfect matchings on i vertices, that is,

$$M(i) = \frac{i!}{(i/2)!2^{i/2}}.$$

(Each time we deal with perfect matchings, i is assumed to be an even number.) After simplification we get

$$X(a, b) = (2n)!(dn/2)!^2 (dn)!(3dn/2 - an - bn)!(dn)!2^{dn} \\ \cdot \left[n!(n/2)!^2 (an)!(dn/2 - an)!(bn)!(dn - bn)!(dn/4 - an/2 - bn/2)! \right. \\ \left. 2^{dn/4 - an/2 - bn/2} (3dn/4 - an/2 - bn/2)! 2^{3dn/4 - an/2 - bn/2} (2dn)! \right]^{-1}.$$

Using Stirling's formula ($i! \approx \sqrt{2\pi i}(i/e)^i$) and focusing on the exponential part we obtain

$$X(a, b) = \Theta(n^{-2})e^{f(a,b,d)n},$$

where

$$f(a, b, d) = (3 - 3d + a + b) \log 2 + d \log d - a \log a - b \log b - (d/2 - a) \log(d/2 - a) \\ - (d - b) \log(d - b) - (d/4 - a/2 - b/2) \log(d/4 - a/2 - b/2) \\ - (3d/4 - a/2 - b/2) \log(3d/4 - a/2 - b/2) \\ + (3d/2 - a - b) \log(3d/2 - a - b).$$

Thus, if $f(a, b, d) \leq -\varepsilon$ (for some $\varepsilon > 0$) for all pairs of integers an and bn under consideration, then we would get $\sum_{an} \sum_{bn} X(a, b) = O(1)e^{-\varepsilon n} = o(1)$ (as $an = O(n)$ and $bn = O(n)$). The desired property would be satisfied, and the proof would be finished.

It is straightforward to see that

$$\frac{\partial f}{\partial b} = \log 2 - \log b + \log(d - b) + \log(d/4 - a/2 - b/2)/2 \\ + \log(3d/4 - a/2 - b/2)/2 - \log(3d/2 - a - b).$$

Now, since $\frac{\partial f}{\partial b} = 0$ if and only if

$$b^2 - b(2d - 2a) + d(d - 2a)/2 = 0,$$

function $f(a, b, d)$ has a local maximum for $b = b_0 := d - a - \sqrt{2d^2 - 4ad + 4a^2}/2$, which is also a global one on $b \in (-\infty, d/2 - a)$. (Observe that since $b \leq d/2 - a$, $b_0 = d/2 - a + d/2 - \sqrt{d^2 + (d - 2a)^2}/2 \leq d/2 - a$.) Consequently,

$$f(a, b, d) \leq g(a, d) := f(a, b_0, d).$$

Finally, let us fix $d_0 = 31$. It is easy to show that $g(a, d_0)$ is an increasing function of a on the interval $0 \leq a \leq a_0 = 1$. Thus, we get $g(a, d_0) \leq g(a_0, d_0) < -0.02 =: -\varepsilon$ and the proof is finished. \square

The above lemma together with Lemma 3.1 (applied with $c = 1$) implies the following.

Theorem 3.6. *A.a.s. $\mathcal{G}_{2n,31} \rightarrow (\mathcal{C}_{\leq n}, P_n)$, which implies that $\hat{R}(\mathcal{C}_{\leq n}, P_n) \leq 31n$ for sufficiently large n .*

4. LOWER BOUNDS

We start with an easy observation. Let \mathcal{C} be a family of all cycles (of any length).

Lemma 4.1. *Let $G = (V, E)$ be a connected graph such that $G \rightarrow (\mathcal{C}, P_n)$. Then,*

$$|E| \geq |V| + n - 2.$$

Proof. Let G be any graph such that $G \rightarrow (\mathcal{C}, P_n)$. Let T be any spanning tree of G . Color the edges of T red and blue otherwise. Clearly, there is no red cycle (of any length). Thus, there must be a blue path on n vertices. This implies that $|V(G)| = |V(T)| \geq n$ and the number of blue edges is at least $n - 1$. Hence, $|E(G)| \geq |V(T)| - 1 + (n - 1)$, and so the result holds. \square

Since obviously if $G \rightarrow (\mathcal{C}, P_n)$, then $|V(G)| \geq n$, we get that Lemma 4.1 implies:

Corollary 4.2. *Let $c \in \mathbb{R}_+$. Then,*

$$\hat{R}(\mathcal{C}_{\leq cn}, P_n) \geq \hat{R}(\mathcal{C}, P_n) \geq 2(n - 1).$$

We will soon improve the leading constant 2. But first, in order to prepare the reader for more complicated argument, we show a weaker result which improves this constant for graphs with bounded maximum degree.

Theorem 4.3. *Let $c \in \mathbb{R}_+$ and let G be a graph with maximum degree Δ such that $G \rightarrow (\mathcal{C}_{\leq cn}, P_n)$. Then,*

$$|E(G)| \geq n(2 + 1/\Delta^2) - 2.$$

Proof. Let $G = (V, E)$ be a connected graph with maximum degree Δ such that $G \rightarrow (\mathcal{C}_{\leq cn}, P_n)$. We will start by showing that $|V| \geq n(1 + 1/\Delta^2)$. Consider G^2 (recall that two vertices are adjacent in G^2 if they are at distance at most 2 in G). Clearly, the maximum degree of G^2 is at most Δ^2 . Moreover, observe that any independent set A in G^2 induces a forest between A and $V \setminus A$ in G (in fact, a collection of disjoint stars as no vertex from $V \setminus A$ is adjacent to more than one vertex from A in G). Finally, clearly there is an independent set A in G^2 of size at least $|V|/(\Delta^2 + 1)$.

Now, let us color the edges of G as follows: color red all edges between A and $V \setminus A$, and blue otherwise. Since the subgraph induced by the edges between A and $V \setminus A$ is a forest, there is no red cycle and so there must be a blue path on n vertices. Such a path must be entirely contained in $G[V \setminus A]$ as $G[A]$ is an empty graph. Thus, $|V \setminus A| \geq n$ and we get

$$|V| = |A| + |V \setminus A| \geq |V|/(\Delta^2 + 1) + n$$

implying that $|V| \geq n(1 + 1/\Delta^2)$, as required.

The rest of the proof is straightforward. We apply Lemma 4.1 to conclude that

$$|E| \geq |V| + n - 2 \geq n(2 + 1/\Delta^2) - 2$$

and the bound holds. \square

Using the ideas from the above proof we improve Corollary 4.2. The improvement of the leading constant might seem negligible. However, it was not clear if one can move away from the constant 2. The observation below answers this question.

Theorem 4.4. *Let $c \in \mathbb{R}_+$. Then for sufficiently large n we have*

$$\hat{R}(\mathcal{C}_{\leq cn}, P_n) \geq \hat{R}(\mathcal{C}, P_n) \geq 2.00365n.$$

Proof. Set $a = 2.0037$, $b = 0.5$, and $d = 9$. Suppose that $G = (V, E)$ is a graph such that $G \rightarrow (\mathcal{C}_{\leq cn}, P_n)$. Clearly, G has at least n vertices. Let us put all vertices of degree at least $d + 1$ to set B . We may assume that B contains at most $2a/(d + 1)$ fraction of vertices; otherwise, G would have more than an edges and we would be done. Let $A \subseteq V \setminus B$ be an independent set in G^2 (and so also in G) as in the proof of Theorem 4.3. That means the graph induced between A and $V \setminus (A \cup B)$ is a collection of disjoint stars. Furthermore, we may assume that A is maximal (that is, no vertex from $V \setminus (A \cup B)$ can be added to A without violating this property). As in the previous proof we notice that A contains at least $(1 - \frac{2a}{d+1})/(d^2 + 1)$ fraction of vertices of G . Now, we need to consider two cases.

Case 1: $|B| \leq b|A|$. Color the edges between A and $V \setminus (A \cup B)$ red and blue otherwise. Clearly, there is no red cycle and so there must be a blue path P_n . Moreover, since $|B| \leq b|A|$, at least $(1 - b)|A|$ vertices are not part of a blue P_n . Thus,

$$|V| \geq n + (1 - b)|A| \geq n + (1 - b) \left(\left(1 - \frac{2a}{d+1}\right) / (d^2 + 1) \right) |V|$$

yielding

$$|V| \geq \left(1 - \frac{(1 - b)(1 - 2a/(d + 1))}{d^2 + 1}\right)^{-1} n.$$

Finally, Lemma 4.1 implies that

$$|E| \geq |V| + n - 2 \geq \left(1 + \left(1 - \frac{(1 - b)(1 - 2a/(d + 1))}{d^2 + 1}\right)^{-1}\right) n - 2 > 2.00366n$$

for sufficiently large n .

Case 2: $|B| > b|A|$. First color the edges between A and $V \setminus (A \cup B)$ red. Then, extend the graph induced by the red edges to maximal forest in $G[V \setminus B]$; remaining edges color blue. Since $A \subseteq V \setminus B$ is maximal, every vertex in $V \setminus (A \cup B)$ has at least one neighbour in A . Thus, the number of red edges is at least $n - |B| - |A|$. As in the previous case there is no red cycle and so there exists a blue P_n . The number of blue edges that are not on such blue P_n is at least $|B|(d + 1 - 2)/2$. Consequently, the total number of edges is at

least

$$\begin{aligned}
 (n - |B| - |A|) + (n - 1) + |B|(d - 1)/2 &\geq 2n - |A| + |B|\frac{d - 3}{2} - 1 \\
 &\geq 2n + |A|\left(\frac{b(d - 3)}{2} - 1\right) - 1 \\
 &\geq \left(2 + \frac{1 - \frac{2a}{d+1}}{d^2 + 1} \left(\frac{b(d - 3)}{2} - 1\right)\right) n - 1 \\
 &> 2.00365n
 \end{aligned}$$

for n large enough. This completes the proof. \square

5. SIZE-RAMSEY OF C_n VERSUS P_n

The lower bound follows immediately from the result obtained by the first and the third author of this paper [9]:

$$\hat{R}(C_n, P_n) \geq \hat{R}(P_n, P_n) \geq 5n/2 - O(1).$$

As a matter of fact this can easily be improved for odd n .

Theorem 5.1. *For all odd positive integers,*

$$\hat{R}(C_n, P_n) \geq 3(n - 1).$$

Proof. Let $G = (V, E)$ be a graph such that $G \rightarrow (C_n, P_n)$. Obviously $|V| \geq n$. Choose an $(n - 1)$ -vertex set $A \subseteq V$ which minimizes the number of edges in the subgraph induced by $V \setminus A$. Color the edges of $G[A]$ blue and the edges between A and $V \setminus A$ red. The remaining edges remain uncolored. Clearly, there is no blue P_n . Furthermore, since n is odd, there is no red C_n . But $G \rightarrow (C_n, P_n)$. Thus, $G[V \setminus A] \rightarrow (K_2, P_n)$ and so $|E(G[V \setminus A])| \geq n - 1$. Now observe that for every $v \in A$, $\deg_{V \setminus A}(v) \geq 2$. Otherwise, if there is a vertex $v \in A$ with $\deg_{V \setminus A}(v) \leq 1$, then we replace v with a vertex $u \in V \setminus A$ of degree 2 in $V \setminus A$ (such vertex u must exist since $G[V \setminus A] \rightarrow (K_2, P_n)$) obtaining smaller number of edges in $G[V \setminus (A \setminus \{v\} \cup \{u\})]$, a contradiction. Consequently,

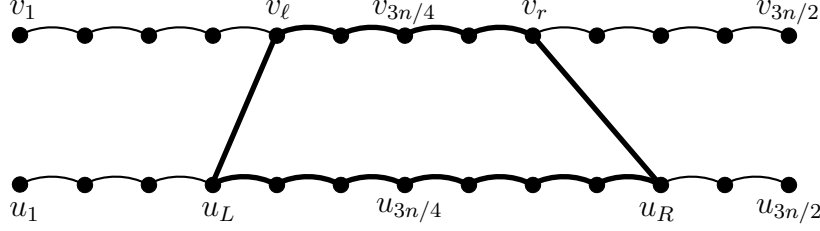
$$|E| \geq 2(n - 1) + (n - 1) = 3(n - 1),$$

and the proof is finished. \square

Now let us focus on the upper bound. We will need the following auxiliary result.

Lemma 5.2 (Corollary 2.1 in [14]). *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph which has no path of length k . Then there exist disjoint subsets $S \subseteq V_1$ and $T \subseteq V_2$ such that $|S| = |T| = (|V_1| + |V_2| - k)/4$ and $e(S, T) = 0$.*

We will estimate now the probability of having a cycle in the union of two random bipartite graphs.

FIGURE 2. Constructing a cycle of length n .

Lemma 5.3. *Let $c > 3/2$ be any constant. Then, $\mathcal{G}(cn, cn, d_1/n) \cup \mathcal{G}(cn, cn, d_2/n)$ fails to have a copy of C_n with probability at most*

$$\exp\left(\left(2c \log c - (c - 3/2) \log\left(\frac{c - 3/2}{2}\right) - (c + 3/2) \log\left(\frac{c + 3/2}{2}\right) - \frac{(c - 3/2)^2 d_1}{4}\right)n\right) + \exp\left(\left((\log 2)/4 - d_2/16\right)n/4\right). \quad (3)$$

Proof. By Lemma 5.2 (applied with $|V_1| = |V_2| = cn$ and $k = 3n$) we obtain that if $\mathcal{G}(cn, cn, d_1/n)$ has no path of length $3n$, then there are two disjoint sets, each of size $(c - 3/2)n/2$, and no edges between them. Thus, the probability that $\mathcal{G}(cn, cn, d_1/n)$ has no P_{3n} is at most

$$\binom{cn}{(c - 3/2)n/2}^2 \left(1 - \frac{d_1}{n}\right)^{\binom{(c - 3/2)n/2}{2}} \leq \exp\left(\left(2c \log c - (c - 3/2) \log\left(\frac{c - 3/2}{2}\right) - (c + 3/2) \log\left(\frac{c + 3/2}{2}\right) - \frac{(c - 3/2)^2 d_1}{4}\right)n\right). \quad (4)$$

Now let us assume that a path $v_1, \dots, v_{3n/2}, u_1, \dots, u_{3n/2}$ of length $3n$ was already found in $\mathcal{G}(cn, cn, d_1/n)$. Let us concentrate on two middle vertices $v_{3n/4}$ and $u_{3n/4}$ that we assume belong to the same partite set, and let us fix an even $i \in [n/4, 3n/4]$. We want to construct a cycle of the desired length as follows: $v_{3n/4}$ to some v_ℓ along the first path ($\ell < 3n/4$), to a specific u_L ($L < 3n/4$), to $u_{3n/4}$ along the second path, continue to u_R for some $R > 3n/4$, to a specific v_r , and go back to $v_{3n/4}$ (see Figure 2). We want the ‘left’ half cycle to be of length i (that is, $3n/4 - \ell + 1 + 3n/4 - L = i$), and the ‘right’ half to be of length $n - i$ (that is, $r - 3n/4 + 1 + R - 3n/4 = n - i$). This guarantees that for different values of i we always investigate disjoint set of edges. The remaining edges of the cycle, that is $\{v_\ell, u_L\}$ and $\{u_R, v_r\}$, will come from $\mathcal{G}(cn, cn, d_2/n)$, independently generated. Furthermore, observe that for a fixed i we have i different choices for $\{v_\ell, u_L\}$. Indeed, we can choose $v_\ell = v_{3n/4-j}$ and $u_L = u_{3n/4-i+j+1}$ for each $0 \leq j \leq i - 1$. Thus, for a given i , the probability that there is no $\{v_\ell, u_L\}$ is at most $(1 - d_2/n)^i$. Similarly, we have $n - i$ choices for $\{u_R, v_r\}$ giving the probability of failure at most $(1 - d_2/n)^{n-i}$. Hence, we fail to find edges $\{v_\ell, u_L\}$ or $\{u_R, v_r\}$ with probability at most

$$(1 - d_2/n)^i + (1 - d_2/n)^{n-i} \leq \exp(-d_2 i/n) + \exp(-d_2(n - i)/n) \leq 2 \exp(-d_2/4),$$

since $i \in [n/4, 3n/4]$. Now, as we have $n/4$ independent events for various values of i (recall that i must be even), we fail to close a cycle with probability at most

$$(2 \exp(-d_2/4))^{n/4} = \exp\left(\left((\log 2)/4 - d_2/16\right)n\right). \quad (5)$$

Thus, by (4) and (5) we get that the probability that $\mathcal{G}(cn, cn, d_1/n) \cup \mathcal{G}(cn, cn, d_2/n)$ contains no copy of C_n is bounded from above by (3). \square

Theorem 5.4. *For all even and sufficiently large n ,*

$$\hat{R}(C_n, P_n) \leq 2257n.$$

Proof. In order to avoid technical problems with events not being independent, we use a classic technique known as *two-round exposure* (known also as *sprinkling* in the percolation literature). The observation is that a random graph $G \in \mathcal{G}(cn, cn, d/n)$ can be viewed as a union of two independently generated random graphs $G_1 \in \mathcal{G}(cn, cn, d_1/n)$ and $G_2 \in \mathcal{G}(cn, cn, d_2/n)$, provided that $d/n = d_1/n + d_2/n - d_1d_2/n^2$ (see, for example, [7, 12] for more information).

Now, consider $\mathcal{G}((2c+1)n, d_1/n) \cup \mathcal{G}((2c+1)n, d_2/n) = \mathcal{G}((2c+1)n, (d_1 + d_2 - o(1))/n)$, and assume that there is no blue P_n . Then, by Lemma 3.2, we get that there are two disjoint sets S and T with $|S| = |T| = cn$ such that all edges between S and T are red. Due to Lemma 5.3 the probability that $G[S, T]$ contains no copy of C_n is at most

$$\exp\left(\left(2c \log c - (c - 3/2) \log\left(\frac{c - 3/2}{2}\right) - (c + 3/2) \log\left(\frac{c + 3/2}{2}\right) - \frac{(c - 3/2)^2 d_1}{4}\right)n\right) + \exp\left(\left((\log 2)/4 - d_2/16\right)n/4\right).$$

On the other hand, the union bound over all choices of S and T contributes only

$$\binom{(2c+1)n}{cn} \binom{(c+1)n}{cn} = \frac{((2c+1)n)!}{(cn)!2n!} = o(1) \cdot \exp\left(\left((2c+1) \log(2c+1) - 2c \log c\right)n\right)$$

number of terms. Since the number of edges present is a.a.s.

$$\binom{(2c+1)n}{2}(d_1 + d_2 - o(1))/n \approx \frac{(2c+1)^2}{2}(d_1 + d_2)n,$$

our goal is to minimize $(2c+1)^2(d_1 + d_2)/2$, provided that

$$(2c+1) \log(2c+1) - (c - 3/2) \log\left(\frac{c - 3/2}{2}\right) - (c + 3/2) \log\left(\frac{c + 3/2}{2}\right) - \frac{(c - 3/2)^2 d_1}{4} \leq 0$$

and

$$(2c+1) \log(2c+1) - 2c \log c + (\log 2)/4 - d_2/16 \leq 0.$$

One can easily check that for $c = 2.21$, $d_1 = 60.34$, and $d_2 = 93.26$ the above inequalities hold and $(2c+1)^2(d_1 + d_2)/2 < 2257$. \square

6. CONCLUDING REMARKS

In this paper we investigated the size-Ramsey number of a collection of forbidden cycles versus a path. In particular, we showed that for n sufficiently large, $2.00365n \leq \hat{R}(\mathcal{C}_{\leq n}, P_n) \leq 31n$. We also considered a more general question and studied $\hat{R}(\mathcal{C}_{\leq cn}, P_n)$ for any positive constant c . For instance we showed that $\hat{R}(\mathcal{C}_{\leq cn}, P_n) \leq \frac{80 \log(e/c)}{c} n$ for fixed $0 < c < 1$. This upper bound is a decreasing function of c . This behavior seems to be correct since in general $\hat{R}(\mathcal{C}_{\leq c_2 n}, P_n) \leq \hat{R}(\mathcal{C}_{\leq c_1 n}, P_n)$ for any $0 < c_1 < c_2$. However, for $c \geq 1$ we only provided a bound that does not depend on c . It is plausible to conjecture that there exists some *decreasing* function $\beta = \beta(c)$ such $\hat{R}(\mathcal{C}_{\leq cn}, P_n) = \beta(c)n + o(n)$ for any fixed $c > 0$. As it was pointed out by one of the referees one can also consider the limiting case when $c \rightarrow \infty$ and study the size-Ramsey number of \mathcal{C} (collection of all cycles) versus a path. Due to Theorem 4.4 we obtain that $\hat{R}(\mathcal{C}, P_n) \geq 2.00365n$. Unfortunately, we were unable to provide any better upper bound than the one based on Theorem 3.6 which yields $\hat{R}(\mathcal{C}, P_n) \leq \hat{R}(\mathcal{C}_{\leq n}, P_n) \leq 31n$.

Acknowledgments We are grateful to the referees for their detailed comments and suggestions that helped us (hopefully) to significantly improve the manuscript.

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