ON SOME MULTICOLOUR RAMSEY PROPERTIES OF RANDOM GRAPHS

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Abstract. The size-Ramsey number $\hat{R}(F)$ of a graph F is the smallest integer m such that there exists a graph G on m edges with the property that any colouring of the edges of G with two colours yields a monochromatic copy of F.

In this paper, first we focus on the size-Ramsey number of a path P_n on n vertices. In particular, we show that $5n/2 - 15/2 \le \hat{R}(P_n) \le 74n$ for n sufficiently large. (The upper bound uses expansion properties of random d-regular graphs.) This improves the previous lower bound, $\hat{R}(P_n) \ge (1+\sqrt{2})n - O(1)$, due to Bollobás, and the upper bound, $\hat{R}(P_n) \le 91n$, due to Letzter.

Next we study long monochromatic paths in edge-coloured random graph $\mathcal{G}(n,p)$ with $pn \to \infty$. Let $\alpha > 0$ be an arbitrarily small constant. Recently, Letzter showed that a.a.s. any 2-edge colouring of $\mathcal{G}(n,p)$ yields a monochromatic path of length $(2/3-\alpha)n$, which is optimal. Extending this result, we show that a.a.s. any 3-edge colouring of $\mathcal{G}(n,p)$ yields a monochromatic path of length $(1/2-\alpha)n$, which is also optimal.

We also consider a related problem and show that for any $r \ge 2$, a.a.s. any r-edge colouring of $\mathcal{G}(n,p)$ yields a monochromatic connected subgraph on $(1/(r-1)-\alpha)n$ vertices, which is also tight.

1. Introduction

Following standard notation, we write $G \to (F)_r$ if any r-edge colouring of G (that is, any colouring of the edges of G with r colours) yields a monochromatic copy of F. For simplicity, we often write $G \to F$ instead of $G \to (F)_2$. Furthermore, we define the *size-Ramsey number* of F as $\hat{R}(F,r) = \min\{|E(G)| : G \to (F)_r\}$, i.e., it is the smallest number of edges in a graph G such that $G \to (F)_r$. Again, for simplicity, we denote $\hat{R}(F) = \hat{R}(F,2)$.

We consider the size-Ramsey number of the path P_n on n vertices. It is easy to see that $\hat{R}(P_n) = \Omega(n)$ and that $\hat{R}(P_n) = O(n^2)$. For example, $K_{2(n-1)} \to P_n$. Indeed, in any 2-colouring of the edges of $K_{2(n-1)}$ every vertex is adjacent to at least $n-1$ edges of the same colour, say red. Furthermore, any two vertices with $n-1$ red adjacent edges are in the same red component. Thus, there is a monochromatic component of order at least n with minimum degree at least $n-1$, which clearly must contain a copy of P_n.

The exact behaviour of $\hat{R}(P_n)$ was not known for a long time. In fact, Erdős [16] offered \$100 for a proof or disproof that

$$\hat{R}(P_n)/n \to \infty \quad \text{and} \quad \hat{R}(P_n)/n^2 \to 0.$$

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This problem was solved by Beck [2] in 1983 who, quite surprisingly, showed that $\hat{R}(P_n) < 900n$. (Each time we refer to inequality such as this one, we mean that the inequality holds for sufficiently large n.) A variant of his proof, provided by Bollobás [11], gives $\hat{R}(P_n) < 720n$. Very recently, the authors of this paper [15] used a different and more elementary argument that shows that $\hat{R}(P_n) < 137n$. The argument was subsequently tuned by Letzter [30] who showed that $\hat{R}(P_n) < 91n$. On the other hand, the first nontrivial lower bound was provided by Beck [3] and his result was subsequently improved by Bollobás [9] who showed that $\hat{R}(P_n) \geq (1+\sqrt{2})n - O(1)$.

In Section 2, we show that for any $r \geq 1$, $\hat{R}(P_n, r) \geq \frac{(r+3)r}{4}n - O(r^2)$ (Theorem 2.1), which slightly improves the lower bound of Bollobás [9] for two colours and generalizes it to more colours. It follows that $\hat{R}(P_n) \geq 5n/2 - O(1)$. In Section 3, using expansion properties of random d-regular graphs, we show that $\hat{R}(P_n) \leq 74n$ (Theorem 3.4) which improves the leading constant provided by Letzter [30]. We also generalize our upper bound to more colours, showing that $\hat{R}(P_n, r) \leq 33r4^r n$ (Theorem 3.6).

In Section 4, we deal with the following, closely related problem. It is known, due to Gerencsér and Gyárfás [22], that $K_n \to P_{(2/3+o(1))n}$; due to Gyárfás, Ruszinkó, Sárközy, and Szemerédi [24, 25] and also Figaj and Łuczak [19], we know that $K_n \to (P_{(1/2+o(1))n})_3$. Moreover, these results are best possible. Unfortunately, very little is known about the behaviour for more colours; although it is conjectured that $K_n \to (P_{(1/(r-1)+o(1))n})_r$ for $r \geq 3$, which would be best possible. Clearly, if for some subgraph G of K_n, $G \to P_{cn}$, then $K_n \to P_{cn}$ as well. On the other hand, one could expect that sparse subgraphs of K_n "arrow" much shorter paths. However, this intuition is false. As a matter of fact, for two colours Letzter [30] showed that a.a.s. $\mathcal{G}(n,p) \to P_{(2/3-\alpha)n}$, provided that $pn \to \infty$, which is optimal. (Here and later on, $\alpha > 0$ is an arbitrarily small constant.) In general, it is known due to Dellamonica, Kohayakawa, Marciniszyn, and Steger [13] that for any $r \geq 3$ a.a.s. $\mathcal{G}(n,p) \to (P_{(1/r-\alpha)n})_r$, provided that $pn \to \infty$. This is, perhaps, not sharp but it is a consequence of the poor current understanding of the behaviour of the multicolored Ramsey number of P_n (see Section 4 for more details). On the other hand, note that the best one can hope for is that a.a.s. $\mathcal{G}(n,p) \to (P_{(1/(r-1)+o(1))n})_r$, provided that $pn \to \infty$, since there are r-colourings of the edges of K_n (and so also of $\mathcal{G}(n,p)$) with no monochromatic path of length $n/(r-1)$. In this paper, we improve the case $r = 3$ and show that a.a.s. $\mathcal{G}(n,p) \to (P_{(1/2-\alpha)n})_3$ (Theorem 4.1), which is optimal.

In the next section, Section 5, we continue with similar direction but relax the property of having P_{cn} as a subgraph to having a component of size cn. It is known, due to Gyárfás [23] and Füredi [21], that for any r-colouring of the edges of K_n, there is a monochromatic component of order $(1/(r-1)+o(1))n$. They also showed that this is best possible if $r-1$ is a prime power. We show that K_n and $\mathcal{G}(n,p)$ behave very similarly with respect to the size of the largest monochromatic component. More precisely, we prove that a.a.s. for any r-colouring of the edges of $\mathcal{G}(n,p)$, there is a monochromatic component of order $(1/(r-1)-\alpha)n$, provided that $pn \to \infty$ (Theorem 5.3). As before, this result is clearly best possible.

2. Lower bound on the size-Ramsey number of P_n

In this section, we improve the lower bound (for two colours) given by Bollobás [9] who showed that $\hat{R}(P_n) \geq (1+\sqrt{2})(n-1)-4$. In our result, the leading constant $(1+\sqrt{2})$ is increased to 5/2. Moreover, we provide a more general result that holds for any number of colours r, which improves the trivial lower bound $\hat{R}(P_n, r) \geq (r-1)(n-1)+1$.

Theorem 2.1. *Let $r \geq 1$. Then, for all sufficiently large n*

$$\hat{R}(P_n, r) \geq \frac{(r+3)r}{4}n - \frac{r(5r+11)}{4} + 3.$$

We will need the following auxiliary claim.

Claim 2.2. *Let $k \geq 0$ and T be a tree. Then, at least one of the following two properties holds:*

(i) T has k edges $e_1, e_2, \ldots, e_k$ such that $T - \{e_1, e_2, \ldots, e_k\}$ contains no P_n,
(ii) T contains $(k+2)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each.

Proof. We prove the statement by induction on k. For $k = 0$, if (i) fails, then T contains a copy of P_n and we are done. Indeed, after splitting the path as equally as possible we get two components of the desired order so (ii) holds.

Let $k \geq 0$ and suppose that the statement holds for any integer i satisfying $0 \leq i \leq k$. Again, assume that (i) fails for $(k+1)$; that is, for any choice of $e_1, e_2, \ldots, e_{k+1}$, $T - \{e_1, e_2, \ldots, e_{k+1}\}$ contains P_n. We will show that (ii) must hold; that is, T contains $(k+3)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each.

Clearly $T \supseteq P_n$. Hence, let e be such that $T - e$ consists of two components, T_1 and T_2, each of order at least $\lfloor n/2 \rfloor$. By the assumption we made (that (i) fails for $(k+1)$), for any choice of k_1 edges $e_1, e_2, \ldots, e_{k_1}$ in T_1 and k_2 edges $f_1, f_2, \ldots, f_{k_2}$ in T_2 such that $k_1 + k_2 = k$, either $T_1 - \{e_1, e_2, \ldots, e_{k_1}\}$ or $T_2 - \{f_1, f_2, \ldots, f_{k_2}\}$ contains P_n.

If $T_1 - \{e_1, e_2, \ldots, e_{k_1}\} \supseteq P_n$ and $T_2 - \{f_1, f_2, \ldots, f_{k_2}\} \supseteq P_n$ for any choice of the edges, then (by inductive hypothesis) T_1 and T_2 have, respectively, (k_1+2) and (k_2+2) vertex-disjoint connected subgraphs of size $\lfloor n/2 \rfloor$, giving $k_1 + k_2 + 4 \geq k+3$ vertex-disjoint connected subgraphs of order $\lfloor n/2 \rfloor$ in T. Therefore, without loss of generality, we may assume that $T_2 - \{f_1, f_2, \ldots, f_{k_2}\} \not\supseteq P_n$ for some choice of $f_1, f_2, \ldots, f_{k_2}$, where k_2 is as small as possible. Of course, this implies that $T_1 - \{e_1, e_2, \ldots, e_{k_1}\} \supseteq P_n$ for any choice of the edges. Now, we need to consider two cases. If $k_2 = 0$, then (by inductive hypothesis) T_1 has (k_1+2) vertex-disjoint connected subgraphs of order $\lfloor n/2 \rfloor$ which, together with T_2 yield $(k+3)$ desired large subgraphs in T. On the other hand, if $k_2 \geq 1$, then (due to minimality of k_2) we infer that for any choice of $f_1, f_2, \ldots, f_{k_2-1}$ we have $T_2 - \{f_1, f_2, \ldots, f_{k_2-1}\} \supseteq P_n$. Thus, (again, by inductive hypothesis) T has $(k_1+2)+(k_2-1+2) = k+3$ vertex-disjoint connected subgraphs of order $\lfloor n/2 \rfloor$, as needed. □

Now, we are ready to prove the main result of this section. The proof uses some ideas of Bollobás [9] and Beck [3].

Proof of Theorem 2.1. We prove the statement by induction on r. For $r = 1$ the desired inequality is trivially true: $\hat{R}(P_n, 1) \geq n-1$. Assume that the statement holds for some

$r \geq 1$ and, for a contradiction, suppose that it fails for $(r+1)$, that is,

$$\hat{R}(P_n, r+1) < \frac{(r+4)(r+1)}{4}n - \frac{(r+1)(5(r+1)+11)}{4} + 3.$$

Let $G = (V, E)$ be a graph of order N and size $\hat{R}(P_n, r+1)$, such that $G \to (P_n)_{r+1}$. Clearly, G is connected. We will independently deal with two cases, depending on N.

Case 1: $N > (r+2)(n-3)/2$. Let T be any spanning tree of G. We apply Claim 2.2 with $k = r$. First, let us assume that property (i) in the claim holds; that is, T has r edges $e_1, e_2, \ldots, e_r$ such that $T - \{e_1, e_2, \ldots, e_r\}$ contains no P_n. We colour all $(N-1) - r$ edges in $T - \{e_1, e_2, \ldots, e_r\}$ using the first colour. The number of uncoloured edges is at most

$$\begin{aligned}\hat{R}(P_n, r+1) &- (N-r-1) \\ &< \frac{(r+4)(r+1)}{4}n - \frac{(r+1)(5(r+1)+11)}{4} + 3 - \frac{r+2}{2}(n-3) + r + 1 \\ &= \frac{(r+3)r}{4}n - \frac{r(5r+11)}{4} + 3 \leq \hat{R}(P_n, r),\end{aligned}$$

where the last inequality follows from the inductive hypothesis. Thus, we can colour the uncoloured edges with the remaining r colours in such a way that there is no monochromatic P_n. Consequently, $G \not\to (P_n)_{r+1}$, which gives us the desired contradiction.

Assume then that property (ii) in the claim holds; that is, T contains $(r+2)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each. We colour $\lfloor n/2 \rfloor - 1$ edges of each of the $(r+2)$ components with the first colour. (If some component has more than $\lfloor n/2 \rfloor - 1$ edges, we select edges to colour arbitrarily.) The number of uncoloured edges is at most

$$\begin{aligned}\hat{R}(P_n, r+1) &- (r+2)\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \\ &< \frac{(r+4)(r+1)}{4}n - \frac{(r+1)(5(r+1)+11)}{4} + 3 - (r+2)\left(\frac{n}{2} - \frac{3}{2}\right) \\ &= \frac{(r+3)r}{4}n - \frac{r(5r+11)}{4} + 3 - (r+1) < \hat{R}(P_n, r),\end{aligned}$$

and this yields a contradiction ($G \not\to (P_n)_{r+1}$), as before.

Case 2: $N \leq (r+2)(n-3)/2$. Let $W_1 \subseteq V$ be any set of size $|W_1| = n-1$, and let $W_2, W_2, \ldots, W_{r+1}$ be an equipartition of $V \setminus W_1$. Clearly, for any $2 \leq i \leq r+1$,

$$|W_i| \leq \left\lceil \frac{1}{r}\left(\frac{r+2}{2}(n-3) - (n-1)\right)\right\rceil = \left\lceil \frac{n-3}{2} - \frac{2}{r} \right\rceil < \frac{n-1}{2} - \frac{2}{r}.$$

For $2 \leq i \leq r+1$, let $G_i = G[W_i, W_1 \cup \cdots \cup W_{i-1}]$ be the bipartite subgraph of G induced by the edges between W_i and $W_1 \cup \cdots \cup W_{i-1}$. We colour the edges of G_i with the i-th colour and the remaining edges (inside W_i's for $1 \leq i \leq r+1$) with the last colour. Clearly there is no monochromatic (or, in fact, any) copy of P_n in W_i's. Furthermore, each path in G_i must alternate between W_i and $W_1 \cup \cdots \cup W_{i-1}$. Thus, the longest path in G_i has at most $2|W_i| + 1 < n$ vertices. We get the desired contradiction ($G \not\to (P_n)_{r+1}$) for the last time and the proof is finished. □

3. Upper bound on the size-Ramsey number of P_n

In this section we improve the upper bound on $\hat{R}(P_n)$ as well as on $\hat{R}(P_n, r)$ for an arbitrary r.

Let us recall a few classic models of random graphs that we study in this section and later on in the paper. The *binomial random graph* $\mathcal{G}(n,p)$ is the random graph G with vertex set $[n] := \{1, 2, \ldots, n\}$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. The *binomial random bipartite graph* $\mathcal{G}(n,n,p)$ is the random bipartite graph $G = (V_1 \cup V_2, E)$ with partite sets V_1, V_2, each of order n, in which every pair $\{i,j\} \in V_1 \times V_2$ appears independently as an edge in G with probability p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity.

Recall that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. Since we aim for results that hold a.a.s., we will always assume that n is large enough. For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible to the asymptomatic calculations we will make. Finally, we use $\log n$ to denote natural logarithms.

However, our main results in this section refer to another probability space, the probability space of random d-regular graphs with uniform probability distribution. This space is denoted $\mathcal{G}_{n,d}$, and asymptotics are for $n \to \infty$ with $d \geq 2$ fixed, and n even if d is odd.

Instead of working directly in the uniform probability space of random regular graphs on n vertices $\mathcal{G}_{n,d}$, we use the *pairing model* (also known as the *configuration model*) of random regular graphs, first introduced by Bollobás [8], which is described next. Suppose that dn is even, as in the case of random regular graphs, and consider dn points partitioned into n labelled buckets $v_1, v_2, \ldots, v_n$ of d points each. A *pairing* of these points is a perfect matching into $dn/2$ pairs. Given a pairing P, we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the buckets $v_1, v_2, \ldots, v_n$, and a pair $\{x, y\}$ in P corresponds to an edge $v_i v_j$ in $G(P)$ if x and y are contained in the buckets v_i and v_j, respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{-(d^2-1)/4}$, so that any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$. For this reason, asymptotic results over random pairings suffice for our purposes. For more information on this model, see, for example, the survey of Wormald [37].

3.1. Existing approach.

Using the following (deterministic) lemma Letzter showed that $\hat{R}(P_n) < 91n$. A similar result was first noticed by Ben-Eliezer, Krivelevich and Sudakov [4, 5] and later by Pokrovskiy [32].

Lemma 3.1 ([4, 5, 30, 32])**.** *Let G be a graph of order cn for some $c > 2$. Assume that for every two disjoint sets of vertices S and T such that $|S| = |T| = n(c-2)/4$ we have $e(S,T) \neq 0$. Then, $G \to P_n$.*

In fact, she showed that a.a.s. $\mathcal{G}(cn, d/n) \to P_n$ with $c = 4.86$ and $d = 7.7$. This is an improved version of a result of the authors of this paper [15]. A slightly stronger bound can be obtained if random d-regular graphs are used. Here a.a.s. $\mathcal{G}_{cn,d} \to P_n$ with $c = 5.219$ and $d = 30$, which implies that $\hat{R}(P_n) < 78.3n$ for sufficiently large n. Since we will provide a stronger result (see Theorem 3.4), we omit the proof.

Lemma 3.1 provides a sufficient condition for $G \to P_n$ that is quite convenient for any good expander G. On the other hand, it is not so difficult to see that it can never give an upper bound better than $26.4n$. Indeed, let $\alpha = (c-2)/(4c)$ and G be a graph of order $N = cn$ and average degree d such that for every two disjoint sets of vertices S and T with $|S| = |T| = \alpha N$ we have $e(S,T) \neq 0$. Then the complement of G contains no copy of $K_{\alpha N, \alpha N}$ and the well-known Kővári, Sós and Turán [28] inequality (see also Theorem 11 in [10]) yields

$$N\binom{N-1-d}{\alpha N} \le (\alpha N - 1)\binom{N}{\alpha N},$$

which for N sufficiently large implies that $d \ge \frac{\log \alpha}{\log(1-\alpha)} - 1$. Thus, the number of edges in G is at least

$$\frac{Nd}{2} = \frac{cnd}{2} \ge \frac{c}{2}\left(\frac{\log \alpha}{\log(1-\alpha)} - 1\right) n = f(c)n,$$

where

$$f(c) := \frac{c}{2}\left(\frac{\log(c-2)/(4c)}{\log(3c+2)(4c) - 1}\right).$$

The above function takes a minimum at $c = c_0 \approx 5.633$ which gives $f(c_0) \approx 26.415$.

3.2. **Improved approach.** In this subsection, we provide another sufficient condition for $G \to P_n$ which can be viewed as a slight straightening of Lemma 3.1. We start with the following elementary observation, which was also first discovered by Ben-Eliezer, Krivelevich and Sudakov [4, 5] and later by Pokrovskiy [32].

Lemma 3.2 ([4, 5, 32])**.** *Let G be a graph of order cn for some $c > 1$. Then, the vertex set $V(G)$ can be partitioned into three sets P, U, W, $|U| = |W| = (cn - |P|)/2$ such that the graph induced by P has a Hamiltonian path and $e(U,W) = 0$.*

Now we are ready to state the main tool used in this subsection.

Lemma 3.3. *Let G be a graph of order cn for some $c > 2$. Assume that for every four disjoint sets of vertices S_1, S_2, T_1, T_2 such that $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$ we have $e(S_1, T_2) \neq 0$ or $e(S_2, T_1) \neq 0$. (Clearly, this implies that $|S_1| = |T_2|$ and $|S_2| = |T_1|$.) Then, $G \to P_n$.*

Proof. Suppose that $G \not\to P_n$; that is, suppose that it is possible to colour the edges of G with the colours blue and red such that there is no monochromatic P_n. Let G_b be the graph on the vertex set $V(G)$, induced by blue edges. It follows from Lemma 3.2 (applied to G_b) that there exist two disjoint sets $U, W \subseteq V(G_b) = V(G)$ each of size $n(c-1)/2$ such that there is no blue edge between U and W (observe that $|P| < n$ as there is no blue P_n in G). Now, consider a bipartite graph $G_r = (U \cup W, E_r)$, with partite sets U, W, and $E_r = \{uw \in E(G) : u \in U, w \in W\}$. Clearly, all edges of G_r are red. Lemma 3.2 (this time applied to G_r) implies then that there exist two disjoint sets $U', W' \subseteq V(G_r) \subseteq V(G)$ each of size $n(c-2)/2$ such that there is no red edge between U' and W' (again, observe that $|P'| < n$ as there is no red P_n in $G \supseteq G_r$). Moreover, as G_r is bipartite, the path P' has at most $n/2$ vertices in U and at most $n/2$ vertices in W. Hence, we may assume that

$|(U' \cup W') \cap U| = |(U' \cup W') \cap W| = n(c-2)/2$. Let $S_1 = U \cap U'$, $S_2 = U \cap W'$, $T_1 = W \cap U'$, and $T_2 = W \cap W'$. Clearly, $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$, $e(S_1, T_2) = 0$, and $e(S_2, T_1) = 0$. The proof of the theorem is finished. □

It is straightforward to show (we omit the proof) that for binomial random graphs with $c = 5.28$ and $d = 6$ a.a.s. $\mathcal{G}(cn, d/n) \to P_n$, which implies that $\hat{R}(P_n) < 83.7n$ for sufficiently large n. As expected, random d-regular graphs give a better constant. Here is the main result of this section.

Theorem 3.4. *Let $c = 5.4806$ and $d = 27$. Then, a.a.s. $\mathcal{G}_{cn,d} \to P_n$, which implies that $\hat{R}(P_n) < 74n$ for sufficiently large n.*

Proof. Consider $\mathcal{G}_{cn,d}$ for some $c \in (2, \infty)$ and $d \geq 1$. Let $s = s(n)$, $a = a(n)$, $b = b(n)$, $t = t(n)$ be functions of n such that sn, an, bn and tn are integers and $0 \leq s \leq (c-2)/4$, $0 \leq a \leq s$, $0 \leq b \leq s$, $0 \leq t \leq \min\{(c-2)/2 - a - b, 2\}$. Let $X(s, a, b, t)$ be the expected number of (ordered) quadruples of disjoint sets S_1, S_2, T_1, T_2 such that $|S_1| = |T_2| = sn$, $|S_2| = |T_1| = ((c-2)/2 - s)n$, $e(S_1, T_2) = e(S_2, T_1) = 0$, $e(S_1, T_1) = adn$, $e(S_2, T_2) = bdn$, and $e(S_1 \cup S_2, V \setminus (S_1 \cup S_2 \cup T_1 \cup T_2)) = tdn$. (Note that, in particular, $|S_1| + |S_2| = |T_1| + |T_2| = |S_1| + |T_1| = |S_2| + |T_2| = n(c-2)/2$.)

Let $M(i)$ be the number of perfect matchings on i vertices, that is,

$$M(i) = \frac{i!}{(i/2)! 2^{i/2}}.$$

(Each time we deal with perfect matchings, i is assumed to be an even number.) Using the pairing model, we get that

$$\begin{aligned} X(s,a,b,t) \quad = \quad & \binom{cn}{sn} \binom{(c-s)n}{(\frac{c-2}{2}-s)n} \binom{\frac{c+2}{2}n}{sn} \binom{(\frac{c+2}{2}-s)n}{(\frac{c-2}{2}-s)n} \binom{sdn}{adn} \binom{(\frac{c-2}{2}-s)dn}{adn} (adn)! \\ & \cdot \binom{sdn}{bdn} \binom{(\frac{c-2}{2}-s)dn}{bdn} (bdn)! \binom{(\frac{c-2}{2}-a-b)dn}{tdn} \binom{2dn}{tdn} (tdn)! \\ & \cdot M\left(\left(\frac{c-2}{2} - a - b - t \right) dn \right) M\left(\left(\frac{c+2}{2} - a - b - t \right) dn \right) / M(cdn). \end{aligned}$$

In this formula the first four binomial coefficients count the number of choices for S_1, T_1, S_2, and T_2. The next factors, $\binom{sdn}{adn}\binom{(\frac{c-2}{2}-s)dn}{adn}(adn)!$, choose adn pairs between S_1 and T_1. Similarly, $\binom{sdn}{bdn}\binom{(\frac{c-2}{2}-s)dn}{bdn}(bdn)!$ deal with bdn pairs between S_2 and T_2. The next three factors, $\binom{\frac{c-2}{2}-a-b)dn}{tdn}\binom{2dn}{tdn}(tdn)!$, count the number of tdn pairs between $S_1 \cup S_2$ and $V \setminus (S_1 \cup S_2 \cup T_1 \cup T_2)$. Finally, $M\left(\left(\frac{c-2}{2} - a - b - t\right) dn\right)$ counts pairs inside $S_1 \cup S_2$, and $M\left(\left(\frac{c+2}{2} - a - b - t\right) dn\right)$ counts the remaining pairs.

Our goal is to show that $X(s, a, b, t) = o(n^{-4})$ (regardless of the choice of s, a, b, t) so that $\sum_{s,a,b,t} X(s, a, b, t) = o(1)$. Hence, we need to maximize $X(s, a, b, t)$. One can show that the maximum is obtained for $a = b$ and for the case when $|S_1| = |S_2| = |T_1| = |T_2| = s = (c-2)/4$. Therefore, we need to concentrate on

$$Y(a,t) = X\left(\frac{c-2}{4}, a, a, t\right) = e^{f(a,t)n + o(n)},$$

where

$$\begin{aligned} f(a,t) &= c\log c + 4(d-1)\left(\frac{c}{4}-\frac{1}{2}\right)\log\left(\frac{c}{4}-\frac{1}{2}\right) + (d-1)2\log 2 - 2da\log a - dt\log t \\ &\quad -\frac{d}{2}c\log c - 4d\left(\frac{c}{4}-\frac{1}{2}-a\right)\log\left(\frac{c}{4}-\frac{1}{2}-a\right) - d(2-t)\log(2-t) \\ &\quad +d\left(\frac{c}{2}-1-2a\right)\log\left(\frac{c}{2}-1-2a\right) - \frac{d}{2}\left(\frac{c}{2}-1-2a-t\right)\log\left(\frac{c}{2}-1-2a-t\right) \\ &\quad +\frac{d}{2}\left(\frac{c}{2}+1-2a-t\right)\log\left(\frac{c}{2}+1-2a-t\right). \end{aligned}$$

Since $\frac{\partial f}{\partial t} = 0$ if and only if $t^2 - (c-4a)t + (c-2-4a) = 0$, function $f(a,t)$ has a local maximum for $t = t_0 := (c-4a)/2 - \sqrt{(c-4a)^2 - 4(c-2-4a)}/2$, which is also a global one on the interval under consideration. We get

$$f(a,t) \le g(a) := f(a,t_0).$$

Finally, by taking $c = 5.4806$ and $d = 27$, we get $g(a) < -0.0001$ for any a we deal with. It follows that for any choice of parameters, $X(s,a,b,t) \le Y(a,t) \le \exp(-0.0001n) = o(n^{-4})$, and the proof is finished. It follows that $\hat{R}(P_n) < 74n$ for n large enough, as $cd/2 = 73.9881 < 74$. □

3.3. **More colours.** In this subsection, we turn our attention to colourings with more than two colours. Here is a natural generalization of Lemma 3.3 in easier, bipartite, setting.

Lemma 3.5. *Let $r \ge 2$ and $G = (V_1 \cup V_2, E)$ be a balanced bipartite graph of order cn for some $c > 2^r - 1$. Assume that for every two sets $S \subseteq V_1$ and $T \subseteq V_2$, $|S| = |T| = ((c+1)/2^r - 1)\,n/2$, we have $e(S,T) \neq 0$. Then, $G \to (P_n)_r$.*

Proof. Suppose that $G \not\to (P_n)_r$; that is, suppose that it is possible to colour the edges of G with the colours from the set $\{1,2,\dots,r\}$ such that there is no monochromatic P_n. Let β_i be defined recursively as follows: $\beta_0 = c$, $\beta_i = (\beta_{i-1}-1)/2$ for $i \ge 1$. Note that $\beta_i = (c+1)/2^i - 1$ for $i \ge 0$. We will use (inductively) Lemma 3.2 to prove the following claim, which will finish the proof (by taking $S = S_r$ and $T = T_r$).

Claim: For each $i \in \{0,1,\dots,r\}$, there exist two sets $S_i \subseteq V_1$ and $T_i \subseteq V_2$, each of size at least $\beta_i n/2$, such that there is no edge between S_i and T_i whose colour belongs to the set $\{1,2,\dots,i\}$.

The base case ($i=0$) trivially (and vacuously) holds by taking $S_0 = V_1$ and $T_0 = V_2$. Suppose that the claim holds for some i, $0 \le i < r$. We apply Lemma 3.2 to the bipartite graph with partite sets S_i, T_i, induced by the edges in colour $(i+1)$. It follows that $S_i \cup T_i$ can be partitioned into three sets P, U, W, P has a Hamiltonian path, $|U| = |W| = (\beta_i n - |P|)/2$, and $e(U,W) = 0$. Since G is bipartite, $|S_i \setminus P| = |T_i \setminus P| = (\beta_i n - |P|)/2$. Without loss of generality, we may assume that $|(S_i \setminus P) \cap U)| \ge |(T_i \setminus P) \cap U)|$. As a result, $|(S_i \setminus P) \cap U)| = |(T_i \setminus P) \cap W)| \ge n(\beta_i - |P|)/4 \ge n(\beta_i - 1)/4$. The inductive step is finished by taking $S_{i+1} = (S_i \setminus P) \cap U$ and $T_{i+1} = (T_i \setminus P) \cap W$. □

Theorem 3.6. *Let $r \ge 2$, $c = 2^{r+1}$, and $d = 8r$. Then, a.a.s. $\mathcal{G}(cn, cn, d/n) \to (P_n)_r$, which implies that $\hat{R}(P_n, r) < 33r4^r n$ for sufficiently large n.*

Proof. Consider $\mathcal{G}(cn, cn, d/n) = (V_1 \cup V_2, E)$. We will show that the expected number of pairs of sets $S \subseteq V_1$ and $T \subseteq V_2$ such that $|S| = |T| = cn/2^{r+2}$ and $e(S,T) = 0$ tends to zero as $n \to \infty$. This will finish the first part of the proof by Lemma 3.5, combined with the first moment principle, as $cn/2^{r+2} < ((c+1)/2^r - 1)\, n/2$ (recall that $c = 2^{r+1}$). Indeed, the expectation we need to estimate is equal to

$$\begin{aligned}\binom{cn}{cn/2^{r+2}}^2 \left(1 - \frac{d}{n}\right)^{(cn/2^{r+2})^2} &\leq (2^{r+2}e)^{2cn/2^{r+2}} \exp\left(-d\left(\frac{c}{2^{r+2}}\right)^2 n\right)\\ &= o\left((e^{2r})^{2cn/2^{r+2}} \exp\left(-d\left(\frac{c}{2^{r+2}}\right)^2 n\right)\right)\\ &= o\left(\exp\left(\left(4r - \frac{dc}{2^{r+2}}\right)\frac{cn}{2^{r+2}}\right)\right) = o(1),\end{aligned}$$

as $dc/2^{r+2} = 4r$. The second part follows from the fact that the number of edges in $\mathcal{G}(cn, cn, d/n)$ is well concentrated around c^2dn and $c^2d = 32r4^r < 33r4^r$. ☐

Summarizing, we showed that there exist some positive constants c_1, c_2 such that for any $r \geq 1$ we have

$$c_1 r^2 \cdot n \leq \hat{R}(P_n, r) \leq c_2 r 4^r \cdot n.$$

Of course, one can improve Lemma 3.5 slightly. For example, in the first step there is no need to assume that the graph is bipartite. Also one could try to use the "double wholes" approach as in Lemma 3.3. However, the improvement would not be substantial. It would be interesting to determine the order of magnitude of $\hat{R}(P_n, r)$ as a function of r (for fixed n).

4. Multicoloured path Ramsey number of $\mathcal{G}(n,p)$

Determining the classical Ramsey number for paths, $R(P_n, r)$, it is a well-known problem that attracted a lot of attention. The case $r = 2$ is well understood, due to the result of Gerencsér and Gyárfás [22]. It is known that

$$R(P_n, 2) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

For $r = 3$ and n sufficiently large, Gyárfás, Ruszinkó, Sárközy, and Szemerédi [24, 25] proved that

$$R(P_n, 3) = \begin{cases} 2n-1 & \text{for odd } n,\\ 2n-2 & \text{for even } n,\end{cases}$$

as conjectured earlier by Faudree and Schelp [18]. (An asymptotic value was obtained earlier by Figaj and Łuczak [19].) However, this problem is still open for *small* values of n. On the other hand, very little is known for any integer $r \geq 4$. The well-known Erdős and Gallai result [17] implies only that $R(P_n, r) \leq rn$. Very recently, Sárközy [33] improved it and showed that for any integer $r \geq 2$,

$$R(P_n, r) \leq \left(r - \frac{r}{16r^3+1}\right) n.$$

It is believed that the value of $R(P_n, r)$ is close to $(r-1)n$, which would be optimal, since for some values of r there are r-colourings of the edges of $K_{(r-1)n}$ with no monochromatic component of size bigger than n (see Section 5).

In this section, we consider an analogous problem for $\mathcal{G}(n,p)$ with average degree, np, tending to infinity as $n \to \infty$. We are interested in the following constant:

$$c_r = \sup\{c \in [0,1] : \mathcal{G}(n,p) \to (P_{cn})_r \text{ a.a.s., provided } np \to \infty\}. \quad (1)$$

The case $r = 2$ is already investigated; due to Letzter [30] we know that $c_2 = 2/3$. For an arbitrary integer $r \geq 3$ it is known due to Dellamonica, Kohayakawa, Marciniszyn, and Steger (see Theorem 7 in [13]) that $c_r \geq 1/r$. Here we improve the case $r = 3$.

Theorem 4.1. *Let $\alpha > 0$ be an arbitrarily small constant and $p = p(n)$ be such that $pn \to \infty$. Then, a.a.s. $\mathcal{G}(n,p) \to \left(P_{(1/2-\alpha)n}\right)_3$, which is optimal. This implies that $c_3 = 1/2$.*

Furthermore, we conjecture that $c_r = n/R(P_n, r)$ for any $r \geq 2$, which is true for $r = 2$ [30] and for $r = 3$, due to the above theorem.

The proof is based on a rather standard application of Sparse Regularity Lemma combined with an ingenious idea of Figaj and Łuczak [19] of "connected matchings".

First we introduce some notation needed to state Sparse Regularity Lemma. For given two disjoint subsets of vertices U and W in a graph G, we define the *p-density* of the edges between U and W as

$$d_p(U,W) = \frac{e(U,W)}{p|U||W|}.$$

Moreover, we say that U, W is an *(ε, p)-regular pair* if, for every $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geq \varepsilon|U|$, $|W'| \geq \varepsilon|W|$, $|d_p(U', W') - d_p(U,W)| \leq \varepsilon$. Suppose that $0 < \eta < 1$, $D > 1$ and $0 < p < 1$ are given. We will say that a graph G is *(η, p, D)-upper-uniform* if for all disjoint subsets U_1 and U_2 with $|U_1| \geq |U_2| \geq \eta|V(G)|$, $d_p(U_1, U_2) \leq D$.

The following theorem, which is a variant of Szemerédi's Regularity Lemma [36] for sparse graphs, was discovered independently by Kohayakawa [27] and Rödl (see, for example, [12]).

Theorem 4.2 (*Sparse Regularity Lemma*)**.** *For every $\varepsilon > 0$, $r \geq 1$ and $D \geq 1$, there exist $\eta > 0$ and T such that for every $0 \leq p \leq 1$, if $G_1, G_2, \dots, G_r$ are (η, p, D)-upper-uniform graphs on the vertex set V, then there is an equipartition of V into s parts, where $1/\varepsilon \leq s \leq T$, for which all but at most $\varepsilon\binom{s}{2}$ of the pairs induce an (ε, p)-regular pair in each G_i.*

There is also an improved version of this lemma given by Scott [34], where the upper-uniform requirement is omitted.

The proof of Theorem 4.1 also relies on the following lemma of Figaj and Łuczak [19].

Theorem 4.3 ([19])**.** *Let $0 < \varepsilon \leq 0.001$ and let G be a graph of order n with at least $(1-\varepsilon)\binom{n}{2}$ edges. Then, for any 3-colouring of the edges of G, there is a monochromatic component which contains a matching saturating at least $(1/2 - 5\varepsilon^{1/7})n$ vertices.*

Proof of Theorem 4.1. Let $\alpha > 0$ and $p = p(n)$ be such that $pn \to \infty$ as $n \to \infty$. We will show that a.a.s. for every 3-edge colouring of $G = \mathcal{G}(n,p) = (V,E)$ there is a monochromatic path of length at least $(1/2 - \alpha)n$.

For each $i \in \{1,2,3\}$, let G_i be a subgraph of G induced by the edges coloured with colour i. Let $0 < \eta < 1$ be any constant. By Chernoff's bound, for any U and W of size at least ηn, the p-density $d_p(U,W)$ in G is at most 2 and so the p-density in each G_i is also at most 2. (Indeed, there are obviously at most $(2^n)^2 = 4^n$ choices for U and W, and for each choice the failure probability is at most $2\exp(-\eta^2 n^2 p/3) = o(4^n)$.) Thus, each G_i is an $(\eta, p, 2)$-upper-uniform graph.

Set $D = 2$ and assume that $\varepsilon > 0$ is sufficiently small. Apply the sparse regularity lemma with above defined ε, D, and $r = 3$. Let η and T be the constants arising from this lemma. Consequently, Theorem 4.2 implies that there is an equipartition of $V = V_1 \cup V_2 \cup \cdots \cup V_s$, where $1/\varepsilon \le s \le T$, for which all but at most $\varepsilon\binom{s}{2}$ of the pairs induce an (ε, p)-regular pair in each G_i.

Let R be the auxiliary (cluster) graph with vertex set $[s]$, where $\{i, j\}$ is an edge if and only if V_i, V_j induce an (ε, p)-regular bipartite graph in each of the 3 colours. Colour $\{i, j\}$ in R by the majority colour appearing between V_i and V_j in G. Again by Chernoff's bound the p-density $d_p(V_i, V_j)$ in G is at least $1/2$. Hence, if $\{i, j\}$ is coloured by c, then $d_p(V_i, V_j)$ in G_c is at least $1/6$.

Observe that the number of edges in R is at least $(1-\varepsilon)\binom{s}{2}$. By Theorem 4.3 we obtain a monochromatic, say red, minimal component F which contains a matching M saturating at least $\ell = (1/2 - 5\varepsilon^{1/7})s$ vertices of R. Let $W = (i_1, i_2, \dots i_k, i_1)$ be a minimal closed walk contained in F which contains M. Clearly, F is a tree and so $k \le 2(s-1)$. We divide each set V_{i_j} into two sets U_j, W_j of equal sizes, that is, $|U_j| = |W_j| = n/(2s)$. For each $e \in M$ we find the first appearance of e in W, say (i_j, i_{j+1}). Let P_e be a longest red path in the bipartite graph $G[U_j, U_{j+1}]$. Since V_{i_j} and $V_{i_{j+1}}$ are (ε, p)-regular with p-density at least $1/6$, Lemma 3.2 implies that P_e covers at least $(1-4\varepsilon)n/s$ vertices of $G[U_j, U_{j+1}]$ for each $1 \le j \le \ell - 1$. Similarly, for the second appearance of e in W we will obtain a red path P'_e that covers at least $(1-4\varepsilon)n/s$ vertices of $G[W_{j+1}, W_j]$. Clearly,

$$\sum_{e \in M} (|P_e| + |P'_e|) \ge 2|M| \cdot (1-4\varepsilon)n/s = (1/2 - 5\varepsilon^{1/7})(1-4\varepsilon)n \ge (1/2 - \alpha/2)n$$

for sufficiently small ε.

Finally using elementary properties of (ε, p)-regular pairs we glue all P_e's and P'_e's (following the order in W) loosing only $O(\varepsilon)n \le \alpha n/2$ vertices. This completes the proof. □

5. Large monochromatic components in $\mathcal{G}(n, p)$

It is easy to see that in every 2-colouring of the edges of K_n there is a monochromatic connected subgraph on n vertices. For three colours the analogue problem was first solved by Gerencsér and Gyárfás [22] (see also [1, 6]). The generalization of this result to any number of colours was proved by Gyárfás [23] and it also follows from a more general result of Füredi [21].

Theorem 5.1 ([23, 21]). *Let $r \ge 2$. Suppose that the edges of K_n are coloured with r colours. Then, there is a monochromatic component with at least $n/(r-1)$ vertices. This result is sharp if $r-1$ is a prime power and $(r-1)^2$ divides n.*

In this section we consider a similar problem for $\mathcal{G}(n, p)$. The following was proven by Spöhel, Steger and Thomas [35] and also independently by Bohman, Frieze, Krivelevich, Loh and Sudakov [7]. Recall that a graph is *r-orientable* if it is possible to direct all of its edges so that the resulting digraph has maximum in-degree at most r.

Theorem 5.2 ([35, 7]). *Let $r \ge 2$ and let τ_r denote the constant which determines the threshold for r-orientability of the random graph $\mathcal{G}(n, rc/n)$. Then, for any constant $c > 0$ the following holds a.a.s.*

(i) If $c < \tau_r$, then there exists an r-colouring of the edges of $\mathcal{G}(n, rc/n)$ in which all monochromatic components have $o(n)$ vertices.
(ii) If $c > \tau_r$, then every r-colouring of the edges of $\mathcal{G}(n, rc/n)$ contains a monochromatic component with $\Theta(n)$ vertices.

Here we complement this result considering the case when the average degree tends to infinity (as $n \to \infty$). This time, we are interested in the following constant:

$$d_r = \sup\{d \in [0,1] \ : \ \mathcal{G}(n,p) \text{ has a monochromatic component on at least } dn \text{ vertices a.a.s., provided } np \to \infty\}.$$

Clearly $d_r \ge c_r$, where c_r is defined as in the previous section (cf. (1)).

Theorem 5.3. *Let $r \ge 2$, $\alpha > 0$ be an arbitrarily small constant, and $p = p(n)$ be such that $pn \to \infty$. Then, a.a.s. for any r-colouring of the edges of $\mathcal{G}(n,p)$ there is a monochromatic component on at least $(1/(r-1) - \alpha)n$ vertices. This implies that $d_r \ge 1/(r-1)$.*

The above theorem together with the sharpness statement of Theorem 5.1 yield that $d_r = 1/(r-1)$ for infinitely many r.

First we derive a perturbed version of Theorem 5.1.

Lemma 5.4. *Let $r \ge 2$ and $0 < \varepsilon \le 1/r^2$. Let G be a graph of order n with at least $(1-\varepsilon)\binom{n}{2}$ edges. Then, for any r-colouring of the edges of G there is a monochromatic component on at least $(1/(r-1) - \varepsilon r^2)n$ vertices.*

Let us note that a special case of this result for $r = 3$ was obtained by Figaj and Łuczak [19]. Our proof is different; we will use the following result of Liu, Morris and Prince [31].

Lemma 5.5 (Lemma 9 in [31])**.** *Let $H = (V_1, V_2, E)$ be a bipartite graph. Assume that $|E| \ge \eta|V_1||V_2|$ for some $\eta > 0$. Then, H has a component on at least $\eta(|V_1| + |V_2|)$ vertices.*

Proof of Lemma 5.4. Let $G = (V, E)$ be a graph of order n with at least $(1-\varepsilon)\binom{n}{2} \ge \binom{n}{2} - (\varepsilon/2)n^2$ edges. For a contradiction, suppose that there is a colouring of the edges of G with r colours so that C, a largest monochromatic component in G, satisfies $|V(C)| < (1/(r-1) - \varepsilon r^2)n$. On the other hand, a simple corollary of the Erdős and Gallai result [17] implies that $|V(C)| \ge (1/r - \varepsilon)n$.

Consider the bipartite graph F induced by the edges of G between $V(C)$ and $V(G) \setminus V(C)$. Clearly, the edges of F are coloured with at most $r-1$ colours (as the colour of C is not used). First observe that

$$\begin{aligned}(\varepsilon/2)n^2 &= |V(C)||V(G) \setminus V(C)| \cdot \frac{\varepsilon n^2}{2|V(C)||V(G) \setminus V(C)|} \\ &\le |V(C)||V(G) \setminus V(C)| \cdot \frac{\varepsilon n^2}{2(1/r - \varepsilon)n \cdot (1 - (1/r - \varepsilon))n}.\end{aligned}$$

Since $\varepsilon \le 1/r^2$ and $r \ge 2$, we get $1/r - \varepsilon = (1 - \varepsilon r)/r \ge (1 - 1/r)/r \ge 1/(2r)$. Thus,

$$(\varepsilon/2)n^2 \le |V(C)||V(G) \setminus V(C)| \cdot \frac{\varepsilon}{2 \cdot 1/(2r) \cdot (r-1)/r} \le |V(C)||V(G) \setminus V(C)| \cdot \varepsilon r^2.$$

Consequently,

$$|E(F)| \ge |V(C)||V(G) \setminus V(C)| - (\varepsilon/2)n^2 \ge (1 - \varepsilon r^2)|V(C)||V(G) \setminus V(C)|.$$

Let H be a subgraph of F induced by the majority colour. Thus,

$$|E(H)| \ge \frac{1}{r-1}(1 - \varepsilon r^2)|V(C)||V(G) \setminus V(C)|,$$

and so Lemma 5.5 implies that there is a monochromatic component of order

$$\frac{1}{r-1}(1 - \varepsilon r^2)n \ge \left(\frac{1}{r-1} - \varepsilon r^2\right)n,$$

that is larger than $|C|$, a largest monochromatic component in G. We get the desired contradiction and the proof is finished. □

Finally, we are ready to sketch the proof of the main result of this section.

Sketch of the proof of Theorem 5.3. This is basically the proof of Theorem 4.1 with Theorem 4.3 replaced by Lemma 5.4. We find a monochromatic spanning tree on $(1/(r-1)-\varepsilon r^2)s$ vertices in the cluster graph, and then we replace each edge by a long path (in a bipartite graph). The union of all the paths forms a connected graph. The sharpness follows immediately from the sharpness of Theorem 5.1. □

6. Concluding remarks

We finish the paper with a few remarks and possible questions for future work. In this paper, we improved both a lower and an upper bound for $\hat{R}(P_n)$ and showed that $5n/2 - O(1) \le \hat{R}(P_n) \le 74n$, but clearly there is still a lot of work that is waiting to be done. Closing the gap is a natural question. However, it seems that in order to obtain a substantial improvement, one needs to develop a new approach to attack this question. For more colours, we proved that $\frac{(r+3)r}{4}n - O(r^2) \le \hat{R}(P_n, r) \le 33r4^r n$. Very recently Krivelevich [29] showed that $\hat{R}(P_n, r)$ is nearly quadratic in r by showing that $\hat{R}(P_n, r) \le r^{2+o_r(1)}n$. In the same paper he also derived a new lower bound by proving that $\hat{R}(P_n, r) \ge (r-2)^2 n - O_r(\sqrt{n})$ assuming that $r \ge 3$ and $r-2$ is a prime power. Observe that for $r \ge 6$ the leading constant $(r-2)^2$ is better than our $\frac{(r+3)r}{4}$.

In this paper, we are also concerned with monochromatic paths and components in $\mathcal{G}(n,p)$, provided that $pn \to \infty$. Exactly the same question can be asked for $\mathcal{G}_{n,d}$. It is known, due to a result of Kim and Vu [26], that if $d \gg \log n$ and $d \ll n^{1/3}/\log^2 n$, then there exists a coupling of $\mathcal{G}(n,p)$ with $p = \frac{d}{n}(1-(\log n/d)^{1/3})$, and $\mathcal{G}_{n,d}$, such that a.a.s. $\mathcal{G}(n,p)$ is a subgraph of $\mathcal{G}_{n,d}$. A recent result of Dudek, Frieze, Ruciński, and Šileikis [14] (see also Section 10.3 in [20]) extends that for denser graphs. Consequently, our results for $\mathcal{G}(n,p)$ model imply immediately the counterpart results for $\mathcal{G}_{n,d}$, provided $d \gg \log n$. It would be interesting to investigate the behaviour for $\Omega(1) = d = O(\log n)$.

Finally, determining the value of c_r (which is the largest constant c such that a.a.s. $\mathcal{G}(n,p) \to (P_{cn})_r$ for $pn \to \infty$) might be of some interest (cf. (1)). Letzter [30] showed that $c_2 = 2/3$ and in this paper we showed that $c_3 = 1/2$. For $r \ge 4$ it is known that $1/r \le c_r \le 1/(r-1)$ but the exact value of c_r still remains unknown. The main barrier in determining c_r is most likely in determining the Ramsey number of P_n.

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