FIGHTING CONSTRAINED FIRES IN GRAPHS

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Abstract. The Firefighter Problem is a simplified model for the spread of a fire (or disease or computer virus) in a network. A fire breaks out at a vertex in a connected graph, and spreads to each of its unprotected neighbours over discrete time-steps. A firefighter protects one vertex in each round which is not yet burned. While maximizing the number of saved vertices usually requires a strategy on the part of the firefighter, the fire itself spreads without any strategy. We consider a variant of the problem where the fire is constrained by spreading to a fixed number of vertices in each round. In the two-player game of k-Firefighter, for a fixed positive integer k, the fire chooses to burn at most k unprotected neighbours in a given round. The k-surviving rate of a graph G is defined as the expected percentage of vertices that can be saved in k-Firefighter when a fire breaks out at a random vertex of G.

We supply bounds on the k-surviving rate, and determine its value for families of graphs including wheels and prisms. We show using spectral techniques that random d regular graphs have k-surviving rate at most $\frac{(1+O(d^{-1/2}))}{k+1}$. We consider the limiting surviving rate for countably infinite graphs. In particular, we show that the limiting surviving rate of the infinite random graph can be any real number in $[1/(k+1), 1]$.

1. Introduction

The *Firefighter Problem* was introduced by Hartnell [15], and is a simplified deterministic model of the spread of fire, diseases, and computer viruses in graphs. All graphs we consider are connected, finite (except in Section 4), simple, and undirected. In *Firefighter*, vertices are either *burning* or not. There is one *firefighter* who is attempting to control the fire. Once a vertex is occupied by the firefighter, it can never burn in any subsequent round, and is called *protected*. The fire begins at some vertex in the first round, and the firefighter chooses some vertex to save. The firefighter can visit any vertex in a given round (for example, he can jump between two non-joined vertices from one round to the next), but cannot protect a vertex on fire. The fire acts without intelligence, and spreads to all non-protected neighbours; such vertices are called *burned*. Once a vertex has been protected, its state cannot change; that is, it can never be on fire. The process stops when the fire can no longer spread. A vertex is *saved* if it is not burned at the end of the process.

Firefighter has been a considered in several familiar graph classes, such as finite and infinite grids [7, 9], cubic graphs [18], and trees [8, 16]. The problem has been studied from both structural and algorithm viewpoints; see the survey [12] for additional background.

We consider a variant of firefighting, called *k-Firefighter* for a fixed positive integer k, where in each round, the fire chooses at most k neighbouring vertices to burn. In k-Firefighter, a move for the fire is to spread to at most k unprotected vertices. The reader should note that this results in a two-player game, where an optimal strategy for the fire aims to burn as

1991 *Mathematics Subject Classification.* 05C57, 05C80, 05C63.

Key words and phrases. Firefighter, surviving rate, random regular graphs, infinite random graph.

Supported by grants from NSERC, MITACS, and Ryerson.

many vertices as possible, and an optimal strategy for the firefighter aims to save as many vertices as possible. By keeping a score between the vertices burned and the vertices saved (on a finite graph), k-Firefighter can also be interpreted as a combinatorial game. We note that the game of k-Firefighter was first suggested as a direction for future investigation in [9].

The game of k-Firefighter considers the spread of fire with limited resources. Rather than spreading to *every* neighbour, the fire is constrained to spread to a fixed number of them. Although this may be appear less natural than the applications of the original Firefighter problem, it has potential applications to the spread of gossip, news, or information in a social network: agents in such networks have finite time and resources, and spread the gossip to a finite selection of friends or followers. For the rest of the paper, k is a fixed positive integer.

We note that unlike Firefighter, where the fire has no strategy, the choice of strategy for the fire is important in k-Firefighter. Consider the graph G in Figure 1, where $m \geq 3$. Suppose the fire breaks out at x and the firefighter protects a. In Firefighter, in the second round the fire spreads to both y and b. The firefighter can then only save two vertices of G. However, in 1-Firefighter if the fire chooses to burns y in the second round, then the firefighter can save all of K_m by protecting b. If the fire burns b rather than y in the second round, then the firefighter can save $\lceil \frac{m}{2} \rceil$ vertices in K_m. We note, however, that in some

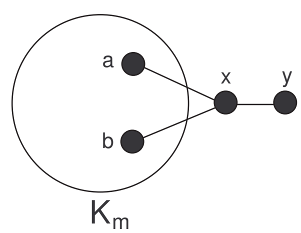


FIGURE 1. The graph G.

graphs (for example, in a clique or a path), the outcome of the game does not depend on how the fire spreads.

The *surviving rate* $\rho(G)$ of a graph G was introduced by Cai and Wang [7] and defined as the expected proportion of vertices that can be saved when a fire breaks out at a vertex of G chosen uniformly at random. Exact values for the surviving rate have been determined for paths and cycles [7] and bounds have been determined for trees [7, 8], planar graphs [7], K_4-free minor graphs [11], and outerplanar graphs [8]. For sparse graphs, surviving rates should be relatively large. Finbow, Wang, and Wang [11] showed that any graph G with $n \geq 2$ vertices and at most $\left(\frac{4}{3} - \varepsilon\right) n$ edges has the property that $\rho(G) \geq \frac{6}{5}\varepsilon$, where $0 < \varepsilon < \frac{5}{24}$ is fixed. In [19] the third author of this paper improved this to show that for graphs with size at most $\left(\frac{15}{11} - \varepsilon\right) n$ has surviving rate $\rho(G) \geq \frac{1}{60}\varepsilon$, where $0 < \varepsilon < \frac{1}{2}$ is fixed. Moreover, a construction of a random graph has been proposed to show that no further improvement is possible; that is, $\frac{15}{11}$ is the threshold (see also [20] for a natural extension of this threshold).

We now define the surviving rate of graphs in the game of k-Firefighter. For a vertex v in G, define $\mathrm{sn_k}(G, v)$ to be the number of vertices that can be saved if a fire breaks out at v.

For a finite graph G, define its *k-surviving rate* to be

$$\rho(G,k) = \frac{1}{n^2} \sum_{u \in V(G)} \mathrm{sn_k}(G,u).$$

Note that $\mathrm{sn_k}(G,u)/n$ is the proportion of vertices that can be saved when a fire breaks out at u. Thus, $\rho(G,k)$ is the expected percentage of vertices that can be saved when a fire breaks out at a random vertex of G (uniform distribution is used for the initial placement). For example, for a clique

$$\rho(K_n,k) = \frac{\lceil (n-1)/(k+1) \rceil}{n} \geq \frac{1}{k+1}\left(1-\frac{1}{n}\right).$$

For a path,

$$\rho(P_n,k) = \rho(P_n) = \frac{2}{n}\cdot\frac{n-1}{n} + \frac{n-2}{n}\cdot\frac{n-2}{n} = 1 - \frac{2}{n} + \frac{2}{n^2}.$$

(To derive these bounds, note that in a clique, the firefighter can save one vertex in each step of the game and then the fire spreads to some k vertices, no matter where the fire breaks out. After $\lceil (n-1)/(k+1) \rceil$ steps the process is finished. For a path, the firefighter can save $n-1$ vertices if the fire breaks out at the end-vertices, and $n-2$ vertices otherwise.)

We focus on the computing the surviving rate for k-Firefighter in various contexts. In Section 2, we supply an upper bound in Theorem 1 for $\rho(G,k)$, where G is a connected graph. In Theorems 2 and 3, we give the exact values of the k-surviving rate for wheels and certain prisms, respectively. We show that random regular graphs have low k-surviving rates for all values of k. It is shown in Theorem 4 that asymptotically almost surely random regular graphs have k-surviving rate at most

$$\frac{1+2d^{-1/2}(\sqrt{k}+O(1))}{k+1} = \frac{(1+O(d^{-1/2}))}{k+1}.$$

where d is the degree of regularity. This tends to $\frac{1}{k+1}$ as d tends to infinity, which is the smallest possible surviving rate in k-Firefighter (see (2.1)). We finish with Section 4, where we introduce the k-surviving rate for infinite graphs as the limit of the k-surviving rate of a chain of finite connected graphs. The definition in the infinite case allows (in certain cases) for different surviving rates. For some graphs, such as cliques or paths, all chains of connected graphs give the same limiting k-surviving rate. We prove in Theorem 9 the surprising result that the k-surviving rate of the infinite random graph can, in fact, be any real number in $[1/(k+1), 1]$.

We emphasize that while the choice by the fire to spread to k neighbours may greatly influence the outcome of the game (as the example in Figure 1 illustrates), in many cases, it does not. Indeed, we use the "intelligence" of the fire in our proofs below explicitly only in Lemma 10.

2. Bounds and graph families

As adding edges does not increase the k-surviving rate, it follows that cliques have the smallest surviving rates. Hence, for a graph with n vertices we have that

$$\rho(G,k) \geq \frac{\lceil (n-1)/(k+1) \rceil}{n} \geq \frac{1}{k+1}\left(1-\frac{1}{n}\right). \tag{2.1}$$

We now derive an upper bound for the k-surviving rate of a connected graph as a function of k and its order.

Theorem 1. *For a connected graph G on n vertices,*

$$\begin{aligned}\rho(G,k) &\leq 1-\frac{2}{n}+\frac{1}{n^2}+\frac{1}{n^2}\left\lceil\frac{n-1}{k+1}\right\rceil. \\ &\leq 1-\frac{1}{n}\left(2-\frac{1}{k+1}\right)+O\Big(\frac{1}{n^2}\Big).\end{aligned} \tag{2.2}$$

Note that the bound in (2.2) is sharp as equality holds for a star on n vertices.

Proof. Since $\rho(G,k)\leq\rho(G-e,k)$ for $e\in E(G)$, we may assume without loss of generality that G is a tree. For a positive integer i, define $[i]=\{1,2,\ldots,i\}$. Let $(d_1,d_2,\ldots,d_n)$ be the degree sequence for T, where $d_i\leq d_{i+1}$ for $i\in[n-1]$. Let v_i be the vertex with degree d_i for $i\in[n]$. If a fire starts at v_i, then the firefighter can save at most one of every $k+1$ neighbours of v_i, thereby saving at most $\lceil\frac{d_i}{k+1}\rceil$ neighbours of v_i. Thus, the fire burns at least $1+\Big(d_i-\lceil\frac{d_i}{k+1}\rceil\Big)$ vertices, which gives the following bound on the proportion of vertices burned:

$$1-\rho(T,k)\geq\frac{1}{n^2}\sum_{i=1}^{n}\Big(1+d_i-\Big\lceil\frac{d_i}{k+1}\Big\rceil\Big).$$

As $\sum_{i=1}^{n}d_i=2(n-1)$, we have that

$$\begin{aligned}\rho(T,k) &\leq 1-\frac{1}{n^2}\Big(3n-2-\sum_{i=1}^{n}\Big\lceil\frac{d_i}{k+1}\Big\rceil\Big) \\ &= 1-\frac{3}{n}+\frac{2}{n^2}+\frac{1}{n^2}\sum_{i=1}^{n}\Big\lceil\frac{d_i}{k+1}\Big\rceil\end{aligned}$$

We leave it as an exercise to show that a degree sequence of trees which maximizes

$$\sum_{i=1}^{n}\Big\lceil\frac{d_i}{k+1}\Big\rceil.$$

is $(1,1,1,\ldots,1,n-1)$. Therefore, we have that

$$\begin{aligned}\rho(T,k) &\leq 1-\frac{3}{n}+\frac{2}{n^2}+\frac{1}{n^2}\Big(1+1+1+\cdots+1+\Big\lceil\frac{n-1}{k+1}\Big\rceil\Big) \\ &= 1-\frac{3}{n}+\frac{2}{n^2}+\frac{n-1}{n^2}+\frac{1}{n^2}\left\lceil\frac{n-1}{k+1}\right\rceil \\ &= 1-\frac{2}{n}+\frac{1}{n^2}+\frac{1}{n^2}\left\lceil\frac{n-1}{k+1}\right\rceil. \quad\square\end{aligned}$$

To further illustrate the parameter $\rho(G,k)$, we now consider exact k-surviving rates for some families of graphs. Let W_n denote the wheel graph, consisting of an $(n-1)$-cycle with all vertices of the cycle joined to a centre vertex.

Theorem 2.

$$\rho(W_n,k)=\begin{cases}1/2 & \text{if } k=1 \text{ and } n=4\\ 1-\frac{4}{n}+\frac{3}{n^2}+\frac{\lceil\frac{n-1}{2}\rceil}{n^2} & \text{if } k=1 \text{ and } n\geq 5\\ 1/4 & \text{if } k\geq 2 \text{ and } n=4\\ 2/5 & \text{if } k=2 \text{ and } n=5\\ 9/25 & \text{if } k>2 \text{ and } n=5\\ 1-\frac{5}{n}+\frac{4}{n^2}+\frac{\lceil\frac{n-1}{k+1}\rceil}{n^2} & \text{if } k\geq 2 \text{ and } n\geq 6\end{cases}$$

Proof. The cases for $n = 4, 5$ are straightforward and so omitted. Assume that $n \geq 6$. If the fire breaks out at one of the $n-1$ vertices of degree three, then the firefighter should protect the centre vertex. What remains for the fire is the cycle C_{n-1}. If $k = 1$, then the firefighter can save all but three vertices on C_{n-1}. If $k = 2$, then the firefighter can save all but four vertices on C_{n-1}. If the fire breaks out at the centre vertex, the firefighter can save exactly one of every $k+1$ neighbours of the centre vertex, saving a total of $\lceil\frac{n-1}{k+1}\rceil$ vertices.

Thus, we have that

$$\rho(W_n,1)=\frac{1}{n^2}\Big((n-3)(n-1)+\Big\lceil\frac{n-1}{k+1}\Big\rceil\Big)=1-\frac{4}{n}+\frac{3}{n^2}+\frac{\lceil\frac{n-1}{2}\rceil}{n^2}$$

$$\rho(W_n,k)=\frac{1}{n^2}\Big((n-4)(n-1)+\Big\lceil\frac{n-1}{k+1}\Big\rceil\Big)=1-\frac{5}{n}+\frac{4}{n^2}+\frac{\lceil\frac{n-1}{k+1}\rceil}{n^2}$$

for $k \geq 2$. □

The *Cartesian product* of G and H, written $G\Box H$, has vertex set $V(G)\times V(H)$. Vertices (a,b) and (c,d) are joined if $a = c$ and $bd \in E(H)$, or $ab \in E(G)$ and $b = d$. We consider *prisms* which are Cartesian products where one of the factors is K_2.

Theorem 3. *For $k \geq 2$, we have the following.*

(1) $\rho(P_n\Box K_2, 1) = 1-\frac{2}{n}+\frac{3}{n^2}$ for $n \geq 4$;
(2) $\rho(P_n\Box K_2, k) = 1-\frac{3}{n}+\frac{7}{n^2}$ for $n \geq 6$;
(3) $\rho(C_n\Box K_2, 1) = 1-\frac{2}{n}$ for $n \geq 4$;
(4) $\rho(C_n\Box K_2, k) = 1-\frac{3}{n}$ for $n \geq 5$.

Proof. The cases for $n = 4, 5$ are trivial and so omitted; assume without loss of generality that $n \geq 6$. In addition, the cases when $k = 1$ (that is, items (1) and (3)) are proved in a manner analogous to the cases when $k \geq 2$, and so are omitted.

First, consider $P_n\Box K_2$. Due to symmetry, there are four possible cases to consider, depending on where the fire breaks out initially. These are described by Figure 3 where the vertices burned are denoted by a black square vertex and the vertices protected are denoted by a white round vertex (the number next to a vertex indicates the round at which the vertex was protected or burned). The initial burned vertex is denoted by the large square.

Figure 2 supplies an optimal strategy for the firefighter. Counting the number of vertices saved in each of the four cases, as well as the number of times each case can occur, we have

$$\begin{aligned}\rho(P_n\Box K_2,k) &= \frac{1}{4n^2}\Big(4(2n-2)+4(2n-4)+4(2n-5)+(2n-12)(2n-6)\Big)\\ &= 1-\frac{3}{n}+\frac{7}{n^2}.\end{aligned}$$

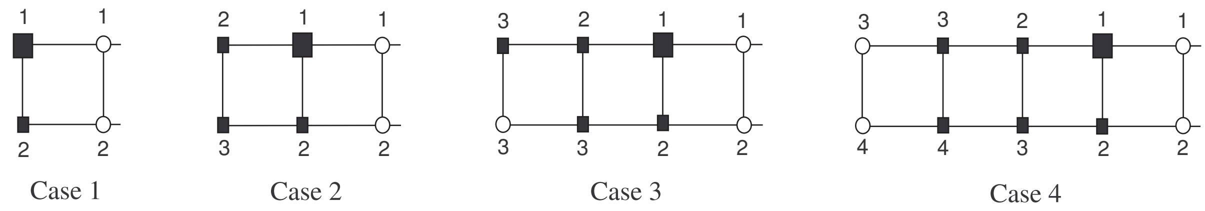


FIGURE 2. The four cases for $P_n \Box K_2$, where $k \geq 2$.

Consider $C_n \Box K_2$ for $n \geq 5$ and $k \geq 2$. Due to symmetry, there is only one possible situation to consider: Case 4. Then $2n-6$ vertices are saved, supplying that

$$\rho(C_n \Box K_2, k) = \frac{1}{4n^2}\Big((2n)(2n-6)\Big) = 1 - \frac{3}{n}. \quad \Box$$

3. Random d regular graphs are flammable

We now consider k-Firefighter played on random d-regular graphs with uniform probability distribution. The probability space is denoted $\mathcal{G}_{n,d}$. An event holds *asymptotically almost surely* or *a.a.s.* in $\mathcal{G}_{n,d}$ if it holds with probability tending to one for $n \to \infty$ with $d \geq 2$ fixed, with the proviso that n even if d is odd. As we will see in Theorem 4, random regular graphs are *flammable*, in the sense that the fire can a.a.s. burn a sizeable proportion of the graph.

Instead of working directly in $\mathcal{G}_{n,d}$, we use the *pairing model* of random regular graphs, first introduced by Bollobás [4]. Suppose that dn is even, as in the case of random regular graphs, and consider dn points partitioned into n labeled buckets $v_1, v_2, \ldots, v_n$ of d points each. A *pairing* of these points is a perfect matching into $dn/2$ pairs. Given a pairing P, we may construct a multigraph $G(P)$, with loops allowed, as follows: the vertices are the buckets $v_1, v_2, \ldots, v_n$, and a pair $\{x, y\}$ in P corresponds to an edge $v_i v_j$ in $G(P)$ if x and y are contained in the buckets v_i and v_j, respectively. It is an easy fact that the probability of a random pairing corresponding to a given simple graph G is independent of the graph, hence the restriction of the probability space of random pairings to simple graphs is precisely $\mathcal{G}_{n,d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{(1-d^2)/4}$ depending on d but not on n. Therefore, any event holding a.a.s. over the probability space of random pairings also holds a.a.s. over the corresponding space $\mathcal{G}_{n,d}$. For this reason, asymptotic results over random pairings suffice for our purposes. One of the advantages of using this model is that the pairs may be chosen sequentially so that the next pair is chosen uniformly at random over the remaining (unchosen) points. For more information on this random graph model, see [21].

The adjacency matrix $A = A(G)$ of a given d-regular graph G with n vertices has n real eigenvalues which we denote by $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Define

$$\lambda = \lambda(G) = \max(|\lambda_2|, |\lambda_n|).$$

Hence, λ is the largest absolute value of an eigenvalue other than d (for more details, see [3, 17]).

We now present an asymptotic upper bound for the k-surviving rate of random d-regular graphs for all values of d and k.

Theorem 4. *Let $d \geq 3$, $k \geq 1$, and fix $\varepsilon > 0$. Let $\lambda = 2\sqrt{d-1} + \varepsilon$. Then, for $G \in \mathcal{G}_{n,d}$ we obtain that a.a.s.*

$$\rho(G,k) \leq \frac{(1+o(1))}{k+1}\left(1+\frac{\lambda}{d}\left(\sqrt{k}+\frac{d}{d-\lambda}\right)\right) = \frac{(1+O(d^{-1/2}))}{k+1}.$$

By (2.1), we have that $\rho(G,k) \to \frac{1}{k+1}$ as $d \to \infty$. Hence, for large values of d, a.a.s. random d-regular graphs have, in a certain sense, the smallest possible k-surviving rate. A stronger (but numerical) result is presented in the next subsection.

Before we prove Theorem 4 we need a few lemmas, the first of which is related to expansion properties of random regular graphs. The value of λ for random d-regular graphs has been studied extensively. A major result due to Friedman [14] is the following.

Lemma 5 ([14])**.** *For every fixed $\varepsilon > 0$ and for $G \in \mathcal{G}_{n,d}$, a.a.s.*

$$\lambda(G) \leq 2\sqrt{d-1} + \varepsilon.$$

The number of edges $|E(S,T)|$ between sets S and T is expected to be close to the expected number of edges between S and T in a random graph of edge density d/n, namely $d|S||T|/n$. A small λ (or large spectral gap) implies that this deviation is small. The following useful bound is essentially proved in [1, 3].

Lemma 6 (Expander Mixing Lemma)**.** *Let G be a d-regular graph with n vertices and set $\lambda = \lambda(G)$. Then for all $S, T \subseteq V$ we have that*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda\sqrt{|S||T|}\,.$$

Note that $S \cap T$ does not have to be empty; in general, $|E(S,T)|$ is defined to be the number of edges between $S \setminus T$ to T plus twice the number of edges that contain only vertices of $S \cap T$. For our purpose here it is better to apply a slightly stronger lower estimate proved in [2, 3] for $|E(S, V\setminus S)|$:

$$|E(S,V\setminus S)| \geq \frac{(d-\lambda)|S||V\setminus S|}{n} \tag{3.1}$$

for all $S \subseteq V$.

The following lemma was essentially proved in [19], although we include a full proof here for completeness. For $d \geq 3$, a cycle is called *short* if it has length at most $L = \log_{d-1}\log_{d-1} n$.

Lemma 7. *If $d \geq 3$ and $G \in \mathcal{G}_{n,d}$, then a.a.s. the number of vertices that belong to a short cycle is at most $\log^2 n$.*

Proof. Let $u \in V(G)$ and let $N_i(u)$ denote the set of vertices at distance at most i from u. A balanced d-regular tree contains d vertices on the first level, $d(d-1)$ vertices on the second level, and so on. Let f_i denote the number of vertices in a balanced d-regular tree with i levels; that is,

$$f_i = 1 + d\sum_{j=0}^{i-1}(d-1)^j = 1 + \frac{d((d-1)^i - 1)}{d-2}.$$

We will show that in the early stages of this process, the graphs grown from u tend to be trees a.a.s.; hence, the number n_i of elements in $N_i(u)$ is equal to f_i a.a.s. In other words, if we expose the vertices at distance $1, 2, \ldots, i$ from u step-by-step, then we have to avoid at step j edges that induce cycles. That is, we wish not to find edges between any two vertices at distance j from u or edges that join any two vertices at distance j to a same vertex at distance $j+1$ from u. We will refer to edges of this form as *special.* Note that the expected number of special edges at step $i+1$ is equal to $O(n_i^2/n) = O(f_i^2/n) = O((d-1)^{2i}/n)$. There are $O(n_i)$ edges created at this point; for a given edge the probability of being special is $O(n_i/m)$. Therefore, the expected number of special edges found up to step $i_1 = \lceil L/2 \rceil$ is equal to

$$\sum_{j=0}^{i_1-1} O\big((d-1)^{2j}/n\big) = O\big((d-1)^{2i_1}/n\big) = O\big(\log n/n\big)\,.$$

(Recall that $L = \log_{d-1}\log_{d-1} n$.) Hence, the expected number of vertices that belong to a cycle of length at most L is $O(\log n)$ and the assertion follows from Markov's inequality. □

We now prove Theorem 4.

Proof of Theorem 4. Consider the vertex set U consisting of vertices from G that do not belong to a short cycle of length at most L. By Lemma 7, since $|U| \geq n - \log^2 n$ a.a.s., we have that a.a.s.

$$\begin{aligned}\rho(G,k) &= \frac{1}{n^2}\sum_{v\in V(G)} \mathrm{sn_k}(G,v)\\ &= \frac{1}{n^2}\sum_{v\notin U}\mathrm{sn_k}(G,v) + \frac{1}{n^2}\sum_{v\in U}\mathrm{sn_k}(G,v)\\ &= o(1) + \frac{1}{n|U|}\sum_{v\in U}\mathrm{sn_k}(G,v).\end{aligned}$$

Therefore, it is enough to show that a.a.s. for all $u \in U$

$$\mathrm{sn_k}(G,u) \leq (1+o(1))\frac{n}{k+1}\left(1+\frac{\lambda}{d}\left(\sqrt{k}+\frac{d}{d-\lambda}\right)\right).$$

(Let us note that when $d = 3$ and $u \notin U$ belongs to the triangle, the firefighter can save the neighbour of u that does not belong to the triangle in the first round, and then apply a greedy strategy of protecting any vertex adjacent to the fire. This strategy saves at least half of vertices, regardless of the value of k. For $k = 1$ it is, in fact, possible to save all but the three vertices that belong to the triangle. Therefore, the desired property might not hold for $u \notin U$, but again, this is not causing any problems since a.a.s. almost all vertices are in U.)

Let $u \in U$ and let B_t, F_t denote the set of vertices burned or protected in round t, respectively. We will examine three successive phases for the fire spreading from u. In the first phase consisting of rounds up to some constant t_0, the fire spreads to all non-protected vertices adjacent to the fire but fewer than k new vertices catch fire in each round. We will use the fact that G is locally a tree to bound t_0. In the second phase defined as those rounds up to a certain $t_1 = t_1(n)$, the fire spreads to k neighbours, but there are still no short cycles. The subgraph induced by B_{t_1} is a tree and there will be $(1+o(1))kt_1$ burned vertices. In the third stage after round t_1, there are short cycles that clearly help the firefighter; however, many vertices are already burned, which makes it impossible for the firefighter to save too many vertices.

In the first phase, in order to minimize $|B_t|$ the firefighter should use a *greedy strategy*; that is, protect a vertex adjacent to the vertex on fire. On the other hand, if $k \geq d$, then the fire cannot use the whole power in this phase and can only spread to less than k vertices in each round. However, this can occur only for a constant number of initial steps. Indeed, in the first round the fire spreads to at least $d-1$ vertices, then to at least $\big((d-1)^2-1\big)$ new ones, and so on. In round t during this phase, there are at least

$$\begin{aligned}&\big((d-1)^t-(d-1)^{t-2}-(d-1)^{t-3}-\cdots-(d-1)-1\big)\\&\qquad=(d-1)^t-\frac{(d-1)^{t-1}-1}{d-2}=\frac{(d-1)^{t-1}(d^2-3d+1)+1}{d-2}\end{aligned}$$

new vertices on fire. Therefore,

$$t_0 \leq \left\lceil \log_{d-1}\left(\frac{k(d-2)-1}{d^2-3d+1}\right)+1\right\rceil,$$

so it is a constant that does not depend on n.

In the second phase, after round t_0, the fire is free to spread to k new vertices in each round. After $t_1 = \lfloor \frac{1}{2}L \rfloor = \lfloor \frac{1}{2}\log_{d-1}\log_{d-1} n \rfloor$ rounds, there are

$$|B_{t_1}| = O(kt_0) + k(t_1 - t_0) = (1+o(1))kt_1$$

vertices on fire that form a tree.

We now consider the third stage after round t_1. It is straightforward to see that the fire will be spreading up to round $\hat{T}$ when $E(B_{\hat{T}}, V\setminus B_{\hat{T}}) = E(B_{\hat{T}}, F_{\hat{T}})$; that is, there is no vertex adjacent to the fire that is not protected. Moreover, if for every $t_0 < t \leq T$ we have that $|E(B_t, V\setminus B_t)| \geq |E(B_t, F_t)| + dk$, the fire spreads with the full speed; that is, $|B_{t+1}| = |B_t| + k$ during this stage of the process. (Since there are at least dk edges from burned to non-protected vertices, there must be at least k non-protected vertices adjacent to the fire.) Hence, $|B_t| = (1+o(1))kt$ during this time period. Note that from (3.1) it follows that for any round t

$$|E(B_t, V\setminus B_t)| \geq \frac{(d-\lambda)|B_t||V\setminus B_t|}{n} = (1+o(1))\frac{(d-\lambda)kt(n-kt)}{n}.$$

Since F_T can receive at most $d|F_T| = dT$ edges, T can be taken to be arbitrarily large as long as T satisfies

$$(1+o(1))\frac{(d-\lambda)kT(n-kT)}{n} \geq dT + dk = (1+o(1))dT.$$

When T is maximized, $T+1$ fails this test which implies that

$$1-\frac{k(T+1)}{n} \leq (1+o(1))\left(\frac{d}{d-\lambda}\right)\frac{1}{k}$$

and so

$$T \geq (1+o(1))\frac{n}{k}\left(1-\left(\frac{d}{d-\lambda}\right)\frac{1}{k}\right). \tag{3.2}$$

A mildly stronger result can be obtained by estimating the number of edges between B_t and F_t using Lemma 6, and inequality (3.2), rough lower bound for T.

$$\begin{aligned} |E(B_t, F_t)| &\leq \frac{d|B_t||F_t|}{n} + \lambda |F_t| \sqrt{\frac{|B_t|}{|F_t|}} \\ &= (1+o(1)) \left(\frac{dkt^2}{n} + \lambda t \sqrt{k} \right). \end{aligned}$$

We therefore have milder condition for T

$$\frac{(d-\lambda)kT(n-kT)}{n} \geq (1+o(1)) \left(\frac{dkT^2}{n} + \lambda T \sqrt{k} \right),$$

and so

$$1 - \frac{(k+1)T}{n} \geq (1+o(1)) \frac{\lambda}{d} \left(1 + \frac{1}{\sqrt{k}} - \frac{kT}{n} \right).$$

Using (3.2), we get that T can be arbitrarily large, provided that T satisfies the following inequality

$$1 - \frac{(k+1)T}{n} \geq (1+o(1)) \frac{\lambda}{d} \left(\frac{1}{\sqrt{k}} + \left(\frac{d}{d-\lambda} \right) \frac{1}{k} \right).$$

As before, when T is taken such that $T+1$ fails this (slightly milder) test, we have that the third phase lasts until time T and

$$T \geq (1+o(1)) \frac{n}{k+1} \left(1 - \frac{\lambda}{d} \left(\frac{1}{\sqrt{k}} + \left(\frac{d}{d-\lambda} \right) \frac{1}{k} \right) \right).$$

Hence, the number of vertices saved $\mathrm{sn_k}(G, u)$ is at most

$$n - (1+o(1))kT \leq (1+o(1)) \frac{n}{k+1} \left(1 + \frac{\lambda}{d} \left(\sqrt{k} + \frac{d}{d-\lambda} \right) \right),$$

and the assertion follows from Lemma 5. □

We note that for the essential elements used in the proof of Theorem 4 are Lemmas 5, 7, and (3.1). The proof will go through in the exact same fashion for a family of random graphs (or a suitably defined family of expander graphs) satisfying these properties. Further, the results of this section hold regardless of how the fire chooses which neighbours to spread.

3.1. **Numerical upper bound.** From Theorem 4 it follows that $\rho(G, k)$ tends to $\frac{1}{k+1}$ as $d \to \infty$. On the other hand, for a given value of d, one can find a better numerical upper bound for $\rho(G, k)$. At the end of the process in round t, there are $|B_t| = kt + O(1)$ vertices on fire, $|F_t| = t$ protected vertices, and $|R_t| = n - (k+1)t + O(1)$ "neutral" vertices that are separated from the fire; that is, there are no edges between B_t and R_t.

Suppose now that for a given $x \in [0, x_0]$ (where $0 \leq x_0 \leq 1$ is a fixed constant) and any $y, z \in [0, 1]$, the expected number $S(x, y, z)$ of partitions of the vertex set into three sets B, F, R of kxn, xn, and $(1-(k+1)x)n$ vertices in $G \in \mathcal{G}_{n,d}$, respectively, with $|E(B, F)| = yn$, $|E(F, R)| = zn$, and $|E(B, R)| = 0$ is $o(n^{-3})$. Then, the expected number of configurations with $x \leq x_0$ is $o(1)$ and so a.a.s. no such partition exists by the first moment method. This implies that a.a.s. the process can last up to round $x_0 n$; that is, a.a.s. $\rho(G, k) < 1 - kx_0$. To find the optimal value of x_0 we use the pairing model.

For any $x \in [0,1]$, it follows that

$$\begin{aligned} S(x,y,z) &= \frac{1}{M(dn)}\binom{n}{kxn}\binom{(1-kx)n}{xn}\binom{kxnd}{yn}\binom{xnd}{yn}(yn)! \\ &\quad \times\binom{xnd-yn}{zn}\binom{(1-(k+1)x)nd}{zn}(zn)! \\ &\quad \times M(kxnd-yn)M(xnd-yn-zn)M((1-(k+1)x)nd-zn), \end{aligned}$$

where $M(i)$ is the number of perfect matchings on i vertices; that is,

$$M(i) = \frac{i!}{(i/2)!2^{i/2}}\,.$$

To see this, we fix kxn vertices ($kxnd$ points) to form set B (the term $\binom{n}{kxn}$), and xn vertices (xnd points) to form set F (the term $\binom{(1-kx)n}{xn}$). Now, we fix yn points in B (the term $\binom{kxnd}{yn}$) that correspond to yn edges that are incident to yn points in F (the term $\binom{xnd}{yn}$). After fixing points in both B and F, we need to connect them in all possible ways (the term $(yn)!$). Similarly, we generate all possibilities for edges from F to R (the term $\binom{xnd-yn}{zn}\binom{(1-(k+1)x)nd}{zn}(zn)!$). Finally, we need to take a perfect matching of remaining points in B (the term $M(kxnd-yn)$), F (the term $M(xnd-yn-zn)$), and R (the term $M((1-(k+1)x)nd-zn)$) to consider all possible configurations satisfying our assumption.

Define

$$g(x,y,z,k,d) = \left(\frac{k^{kx(d-1)}x^{(k+1)x(d-1)}d^{d/2}(1-(k+1)x)^{(1-(k+1)x)(d-1)}}{y^y z^z (kxd-y)^{(kxd-y)/2}(xd-y-z)^{(xd-y-z)/2}((1-(k+1)x)d-z)^{(1-(k+1)x)d-z)/2}}\right)^n.$$

After simplification, using Stirling's formula (that is, $n! \sim \sqrt{2\pi n}(n/e)^n$), and taking the exponential part we obtain that

$$\begin{aligned} S(x,y,z) &\leq e^{o(n)}g(x,y,z,k,d) \\ &= e^{f(x,y,z,k,d)n+o(n)}, \end{aligned}$$

where

$$\begin{aligned} f(x,y,z,k,d) &= kx(d-1)\ln k + (k+1)x(d-1)\ln x + \frac{d}{2}\ln d + (1-(k+1)x)\ln(1-(k+1)x) \\ &\quad -y\ln y - z\ln z - \frac{1}{2}(kxd-y)\ln(kxd-y) - \frac{1}{2}(xd-y-z)\ln(xd-y-z) \\ &\quad -\frac{1}{2}((1-(k+1)x)d-z)\ln((1-(k+1)x)d-z)\,. \end{aligned}$$

It follows that if $\max_{y,z} f(x,y,z,k,d) < 0$, then $S(x,y,z)$ is exponentially small (for n large) for any y,z. To maximize the function f, we need to consider the following system of partial differential equations:

$$\begin{aligned} 0 = \frac{\partial f}{\partial y} &= -\ln y + \frac{1}{2}\ln(kxd-y) + \frac{1}{2}\ln(dx-y-z) \\ 0 = \frac{\partial f}{\partial z} &= -\ln z + \frac{1}{2}\ln(dx-y-z) + \frac{1}{2}\ln((1-(k+1)x)d-z). \end{aligned}$$

We therefore obtain that

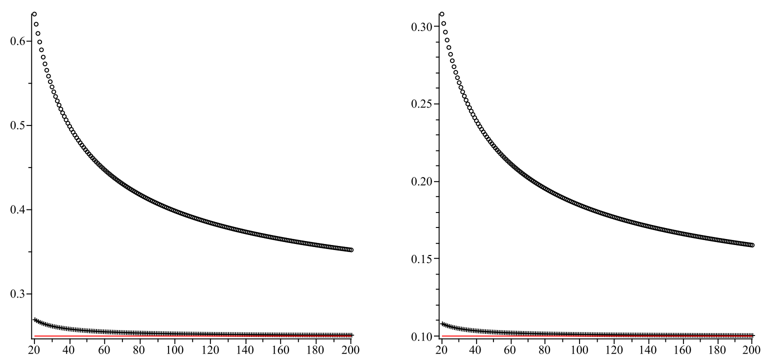
$$\frac{y_0^2}{kxd-y_0} = \frac{z_0^2}{(1-(k+1)x)d-z_0} = dx-y_0-z_0,$$

which can be solved numerically. (Note that y_0 and z_0 are functions of x, as well as of k and d.) Finally, if

$$f(x, y_0, z_0, k, d) = f(x, y_0(x, k, d), z_0(x, k, d), k, d) = f(x, k, d) < 0$$

for every $x \in [0, x_0]$, we derive that a.a.s. $\rho(G, k) < 1 - kx_0$.

We used this approach to obtain asymptotically almost sure upper bounds $u(d, k)$ for the surviving rate of random d-regular graph. When we quote numerical values for $u(d, k)$, we use five decimal places rounded up for upper bounds. For comparison purposes, let $\bar{u}(d, k)$ denote an explicit upper bound following from Theorem 4. In Figure 3, the values of $u(d, k)$ and $\bar{u}(d, k)$ are given for $k = 3$ and $k = 9$ and all d-values between 20 and 200; we also listed the first 18 and a few more values for higher d in Tables 1 and 2. The computations presented in the paper were performed using C/C++. The code may be found on-line at [6].



(a) $k = 3$ (b) $k = 9$

FIGURE 3. Numerical and explicit upper bounds for d from 20 to 200.

d	$u(d,3)$	$\bar{u}(d,3)$	d	$u(d,3)$	$\bar{u}(d,3)$	d	$u(d,3)$	$\bar{u}(d,3)$
3	0.92591	4.77957	12	0.28945	0.79836	25	0.26517	0.58084
4	0.56484	2.24103	13	0.28523	0.76604	50	0.25684	0.46847
5	0.43280	1.59642	14	0.28180	0.73859	75	0.25440	0.42376
6	0.37460	1.30452	15	0.27897	0.71492	100	0.25324	0.39828
7	0.34316	1.13598	16	0.27659	0.69425	125	0.25257	0.38135
8	0.32383	1.02483	17	0.27456	0.67600	150	0.25212	0.36908
9	0.31088	0.94519	18	0.27281	0.65974	175	0.25181	0.35966
10	0.30165	0.88481	19	0.27129	0.64513	200	0.25158	0.35215
11	0.29477	0.83715	20	0.26996	0.63193			

TABLE 1. Numerical and explicit upper bounds for $k = 3$ for some d values.

4. k-SURVIVING RATES OF THE INFINITE RANDOM GRAPH

Firefighter is often studied in the context of infinite grid graphs in the plane. However, in countably infinite graphs, we cannot define the k-surviving rate in the exact same fashion as for finite graphs. For example, we cannot divide by the order of an infinite graph. A

d	$u(d,9)$	$\bar{u}(d,9)$	d	$u(d,9)$	$\bar{u}(d,9)$	d	$u(d,9)$	$\bar{u}(d,9)$
3	0.40646	2.03138	12	0.11606	0.38944	25	0.10628	0.28203
4	0.21998	1.00622	13	0.11438	0.37399	50	0.10285	0.22289
5	0.17029	0.74001	14	0.11301	0.36075	75	0.10183	0.19859
6	0.14866	0.61632	15	0.11187	0.34923	100	0.10135	0.18455
7	0.13683	0.54313	16	0.11091	0.33909	125	0.10107	0.17514
8	0.12946	0.49380	17	0.11009	0.33007	150	0.10089	0.16827
9	0.12446	0.45777	18	0.10939	0.32199	175	0.10076	0.16298
10	0.12086	0.43001	19	0.10877	0.31468	200	0.10066	0.15875
11	0.11816	0.40777	20	0.10823	0.30804			

TABLE 2. Numerical and explicit upper bounds for $k = 9$ for some d values.

natural definition is to write the graph as a limit of finite graphs, and take the limit of k-surviving rate of graphs in the chain. However, different chains can lead to different limiting k-surviving rates.

Let $(G_t : t \in \mathbb{N})$ be a *chain* of finite connected graphs; that is, the sets $\{V(G_t) : t \in \mathbb{N}\}$ and $\{E(G_t) : t \in \mathbb{N}\}$ are well-ordered. Define

$$V(G) = \bigcup_{t\in\mathbb{N}} V(G_t), \qquad E(G) = \bigcup_{t\in\mathbb{N}} E(G_t).$$

We write $\lim_{t\to\infty} G_t = G$, and say that G is the *limit of the chain* $(G_t : t \in \mathbb{N})$. For example, an infinite one way path (or *ray*) is the limit of the chain $(P_t : t \in \mathbb{N})$.

Let $\mathcal{C} = (G_t : t \in \mathbb{N})$ be a chain of finite graphs (note that the limit of $\mathcal{C}$ is connected as each G_t is). Define the *limiting k-surviving rate of G (relative to the chain $\mathcal{C}$)* by

$$\rho_{\mathcal{C}}(G, k) = \lim_{t\to\infty} \rho(G_t, k),$$

assuming this limit exists. Observe that $\rho_{\mathcal{C}}(G, k)$, when it exists, is a real number in the interval $[1/(k+1), 1]$. The value $\rho_{\mathcal{C}}(G, k)$ may strongly depend on the chain $\mathcal{C}$ used, but not in all cases. For example, for a ray $P = \lim_{t\to\infty} P_t$, every chain is a set of paths, and so $\rho_{\mathcal{C}}(P, k) = 1$. The countably infinite clique has surviving rate $\frac{1}{k+1}$ regardless of the chain used (each element of the chain is a clique).

Define the probability space $G(\mathbb{N}, p)$ to be graphs with vertex set of positive integers, and each distinct pair of integers is joined independently with probability $p \in (0, 1)$. We will call this space *the infinite random graph.* Erdős and Rényi [10] discovered the following theorem.

Theorem 8. *For $p \in (0, 1)$ with probability 1, the graph $G(\mathbb{N}, p)$ is unique up to isomorphism.*

Define a graph to be *existentially closed* or *e.c.* if all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex z joined to all of A and to no vertex of B. We say that z is *correctly joined to* A and B. The proof of Theorem 8 follows by proving that with probability 1, $G(\mathbb{N}, p)$ is e.c., and then proving that any two e.c. graphs are isomorphic.

The unique isomorphism type of infinite random graph is written R. We exploit the following explicit representation of R as a limit graph. For a graph H, for each non-empty subset S of $V(H)$, add a new vertex z_S joined to S and to no other vertices. The resulting graph, written $H(1)$, contains H as an induced subgraph. Note that if H has order n, then the order of $H(1)$ is $n + 2^n - 1$ (note that we omit the case with $S = \emptyset$ as that would introduce an isolated vertex). We may iterate this process, so $H(t+1) = (H(t))(1)$. It is easy to see that for a finite graph H, $\lim_{t\to\infty} H(t)$ is e.c., and so is isomorphic to R. (Choose t large enough such that $H(t)$ contains both A and B. A vertex correctly joined to A and B may be found in $H(t+1)$.)

Our main result in this final section is that for the infinite random graph, we can obtain any real k-surviving rate in $[1/(k+1), 1]$.

Theorem 9. *For each real number $r \in [\frac{1}{k+1}, 1]$, there is a chain $\mathcal{C} = (G_t : t \in \mathbb{N})$ of finite graphs such that*

(1) $\lim_{t\to\infty} G_t \cong R$.
(2) $\rho_{\mathcal{C}}(R, k) = r$.

We note that an analogous result was obtained for the cop number of a graph in [5], but with one important caveat. In the definition of so-called *cop density* of a countably infinite graph relative to a chain of finite induced subgraphs (that is, the limit of the ratio of the cop number to the order of the graphs in the chain), graphs in the chain were allowed to be *disconnected*. By a result of Frankl [13], if the graphs in the chain are connected, then the limiting cop density is always 0 (he proved that the cop number of a connected graph of order n is $O(n\frac{\log\log n}{\log n}) = o(n)$.) Hence, Theorem 9 stands in stark contrast to the analogous cop density theorem for R.

To prove the theorem, we need the following two lemmas.

Lemma 10. *Fix a finite graph G of order n. Then,*

$$\rho(G(1), k) = \frac{1+o(1)}{k+1}.$$

Lemma 11. *Fix c a non-negative real number, a graph G of order N, and a vertex v of G. Define $G(1)^{(c,v)}$ by first forming $G(1)$ of order $n = N + 2^N - 1$, then attaching a path of length $\lceil cn \rceil$ to v. Then*

$$\rho(G(1)^{(c,v)}, k) = \frac{\frac{1}{k+1} + c(c+2)}{(c+1)^2} + o(1).$$

Note that if $c \to 0$, then

$$\frac{\frac{1}{k+1} + c(c+2)}{(c+1)^2} \to \frac{1}{k+1},$$

while $c \to \infty$ implies that

$$\frac{\frac{1}{k+1} + c(c+2)}{(c+1)^2} = \frac{\frac{1}{k+1}\frac{1}{c^2+2c} + 1}{1 + \frac{1}{c^2+2c}} \to 1.$$

In particular, Lemma 11 tells us that by choice of c, we can make $\rho(G(1)^{(c,v)}, k)$ as close as we like to any fixed real in $[\frac{1}{k+1}, 1]$.

Proof of Theorem 9. Fix a sequence c_t of non-negative reals such that

$$\lim_{t\to\infty} \frac{\frac{1}{k+1} + c_t(c_t+2)}{(c_t+1)^2} = r.$$

Note that if $r < 1$, then one can take a constant sequence. In order to get $r = 1$, we need to take any sequence tending to infinity.

Fix $G_0 = K_1$. For $t \geq 0$, define $G_{t+1} = (G_t(1))^{(c_t,v)}$, where v is chosen arbitrarily in $G_t(1)$. Note that $(G_t : t \in \mathbb{N})$ forms a chain $\mathcal{C}$, and $G = \lim_{t\to\infty} G_t$ is e.c. and so $G \cong R$.

By Lemma 11,

$$\rho(G_{t+1}, k) = \frac{\frac{1}{k+1} + c_t(c_t+2)}{(c_t+1)^2} + o(1).$$

Hence, $\rho_{\mathcal{C}}(R, k) = r$. □

Proof of Lemma 10. We call vertices in $V(G(1))\backslash V(G)$ *new*. The new vertices with degree one are called *bad*, and the remaining new vertices are *good*. Note that there are n bad vertices and so there are $2^n - 1 - n$ good vertices.

We consider cases of where the fire can break out. If a fire breaks out on G, then each vertex of G is joined to n new vertices of degree $n-1$ so the fire can reach at least one such a vertex in the next round. Spreading a fire from this vertex back to G, the firefighter can only save at most $(1+o(1))\frac{n}{k+1}$ of these. Now a fire spreads to new vertices. Since all but $2^{(1+o(1))n/(k+1)} = o(2^n)$ new vertices are adjacent to the fire, the firefighter can only save at most $(1+o(1))\frac{2^n}{k+1}$ new vertices. If a fire starts at a good new vertex, then at least one vertex of G can be set on fire in the next round. Hence, in the remaining rounds we are back in the first case. If a fire breaks out on a bad vertex, then all but one vertex of $G(1)$ can be saved.

We therefore have that

$$\begin{aligned} \rho(G(1), k) &\leq \frac{n + 2^n - 1 - n}{n + 2^n - 1} \cdot \frac{(1+o(1))\frac{1}{k+1}(n+2^n)}{(n+2^n-1)} + \frac{n}{n+2^n-1} \cdot \frac{n+2^n-2}{n+2^n-1} \\ &= \frac{1+o(1)}{k+1}. \end{aligned}$$

The proof of the lemma now follows from (2.1). □

Proof of Lemma 11. By Lemma 10, $\rho(G(1), k) = \frac{1+o(1)}{k+1}$. If a fire breaks out on $G(1)$, then the firefighter chooses to save v in the first round (or save the vertex on the path adjacent to v in the case v is on fire). Then $(1+o(1))\frac{n}{k+1}$ vertices can be saved in G by protecting any vertex in $G(1)$ regardless of the fire's strategy. As the fire cannot spread to the path, an additional $(1+o(1))cn$ vertices are saved. Note that this strategy is asymptotically the best possible; if the firefighter decides not to save v in the first round and focus on protecting graph $G(1)$, he would not be able to save more than $(1+o(1))\frac{n}{k+1}$ vertices in $G(1)$ by Lemma 10.

If the fire breaks out on a vertex of the path not in $G(1)$, then the firefighter can contain it to save all but at most two vertices in $G(1)^{(c,v)}$. Hence,

$$\begin{aligned} \rho(G(1)^{(c,v)}, k) &= (1+o(1))\frac{n\left(\frac{n}{k+1} + cn\right) + cn(cn+n)}{(cn+n)^2} \\ &= (1+o(1))\frac{\left(\frac{1}{k+1} + c\right) + (c^2+c)}{(c+1)^2} \\ &= \frac{\frac{1}{k+1} + c(c+2)}{(c+1)^2} + o(1). \quad \square \end{aligned}$$

5. Acknowledgements

We thank the anonymous referees whose comments and corrections greatly improved the paper.

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