Vertex Pursuit Games in Stochastic Network Models

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Dedicated to the memory of Aubrey C. Hamlyn.

Abstract. Random graphs with given expected degrees $G(\mathbf{w})$ were introduced by Chung and Lu so as to extend the theory of classical $G(n, p)$ random graphs to include random power law graphs. We investigate asymptotic results for the game of Cops and Robber played on $G(\mathbf{w})$ and $G(n, p)$. Under mild conditions on the degree sequence $\mathbf{w}$, an asymptotic lower bound for the cop number of $G(\mathbf{w})$ is given. We prove that the cop number of random power law graphs with n vertices is asymptotically almost surely $\Theta(n)$. We derive concentration results for the cop number of $G(n, p)$ for p as a function of n.

1 Introduction

Vertex pursuit games, such as Cops and Robber, may be viewed of as a simplified model for network security. As a general motivation for these games, suppose that an intruder (the robber) is loose on a network, and travels between adjacent vertices in an effort to escape the authorities (the cops). The intruder could be a virus or hacker, or some other malicious agent. The goal is to minimize the resources (that is, number of cops) required to capture the intruder.

The game of *Cops and Robber*, introduced independently by Nowakowski and Winkler [9] and Quilliot [10] over twenty years ago, is played on a fixed graph G. We will assume in this paper that G is undirected, simple, and finite. There are two players, a set of k *cops* (or *searchers*), where $k > 0$ is a fixed integer, and the *robber*. The

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cops begin the game by occupying a set of k vertices. The robber then chooses a vertex, and the cops and robber move in alternate rounds. The players use edges to move from vertex to vertex. More than one cop is allowed to occupy a vertex, and the players may remain on their current vertex. The players know each others current locations and can remember all the previous moves. The cops win and the game ends if at least one of the cops can eventually occupy the same vertex as the robber; otherwise, the robber wins. As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, which is the minimum number of cops needed to win on G. The cop number was introduced by Aigner and Fromme [1] who proved that if G is planar, then $c(G) \leq 3$.

So-called *cop-win* graphs (that is, graphs G with $c(G) = 1$) were structurally characterized in [9, 10]. See Figure 1 for a cop-win graph. If x is a vertex, then define $N[x]$ to be x along with the vertices joined to x. The cop-win graphs are exactly those graphs which are *dismantlable*: there exists a linear ordering $(x_j : 1 \leq j \leq n)$ of the vertices so that for all $2 \leq j \leq n$, there is a $i < j$ such that $N[x_j] \subseteq N[x_i]$. No analogous structural characterization of graphs with cop number k, where $k > 1$ is a fixed integer, is known; this is a central open problem in the subject. For a survey of results on the cop number and related search parameters for graphs, see [2].

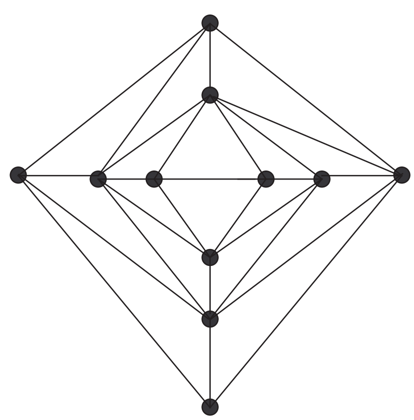

Fig. 1. A cop-win graph.

In the last few years there was an explosion of mathematical research related to stochastic models of real-world networks, especially for models of the web graph. Many technological, social, biological networks have properties similar to those present in the web, such as power law degree distributions and the small world property. We refer to these networks as *self-organizing*. For example, power laws have been observed in protein-protein interaction networks, and networks formed by scientific collaborators. While much of the earlier mathematical work on self-organizing networks focused on designing models satisfying certain properties such as power law degree distributions, new approaches are constantly emerging.

We study vertex pursuit games in models for stochastic network models in self-organizing networks. To our best knowledge, our work is the first to consider such games in these network models. We consider Erdős, Rényi $G(n,p)$ random graphs and their generalizations used to model self-organizing networks. Define a probability space on graphs of a given order $n \geq 1$ as follows. Fix a vertex set V consisting of n distinct elements, usually taken as $[n] = \{1, 2, \ldots, n\}$, and fix $p \in [0,1]$. Note that p can be a function of n. Define the space of *random graphs of order n with edge probability p*, written $G(n,p)$, with sample space equalling the set of all $2^{\binom{n}{2}}$ (labelled) graphs with vertex set V, and

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

Informally, we may view $G(n,p)$ as of graphs with vertex set V, so that two distinct vertices are joined independently with probability p.

The cop number of $G(n,p)$ was studied in [3], where the following result was proved. In this paper, all asymptotics are as $n \to \infty$. We say that an event holds *asymptotically almost surely* (*a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity.

Theorem 1. *Let $0 < p < 1$ be fixed. For every real $\varepsilon > 0$ a.a.s. for $G \in G(n,p)$*

$$(1-\varepsilon)\log_{\frac{1}{1-p}} n \leq c(G) \leq (1+\varepsilon)\log_{\frac{1}{1-p}} n.$$

Recent work of Chung and Lu [5, 6] supplies an extension of the $G(n, p)$ random graphs to random graphs with given expected degree sequence $\mathbf{w}$. The corresponding probability space is referred to as $G(\mathbf{w})$. For example, if $\mathbf{w}$ follows a power law distribution, then $G(\mathbf{w})$ supplies a model for self-organizing networks. We will define $G(\mathbf{w})$ precisely in the next section.

The results in this paper are divided into two parts: bounding the cop number of random graphs with given expected degree and random power law graphs (Section 2), and the cop number of $G(n, p)$ random graphs where p is a function of n (Section 3). Our approach in both sections is to exploit dominating sets to give upper bounds for the cop number, while lower bounds usually follow by considering certain adjacency properties. In random power law graphs, we prove in Theorem 3 that the cop number is $\Theta(n)$.

2 The cop number in random graphs with given expected degree sequence

Let

$$\mathbf{w} = (w_1, \ldots, w_n)$$

be a sequence of n real nonnegative real numbers. We define a random graph model, written $G(\mathbf{w})$, as follows. Vertices are integers in $[n]$. Each potential edge between i and j is chosen independently with probability $p_{ij} = w_i w_j \rho$, where

$$\rho = \frac{1}{\sum_{i=1}^{n} w_i}.$$

We will always assume that

$$\max_i w_i^2 < \sum_{i=1}^{n} w_i,$$

which implies that $p_{ij} \in [0, 1)$. The model $G(\mathbf{w})$ is referred to as *random graphs with given expected degree sequence* $\mathbf{w}$. Observe that $G(n, p)$ may be viewed as a special case of $G(\mathbf{w})$ by taking $\mathbf{w}$ to be equal the constant n-sequence $(pn, pn, \ldots, pn)$.

In our main result of this section, we supply an asymptotic lower bound for the cop number of graphs $G \in G(\mathbf{w})$ which generalizes the lower bound from Theorem 1. Our results demonstrate that a logarithmic lower bound is ubiquitous in random graphs with given expected degrees satisfying our conditions. Let $M = \max_i w_i$ and $m = \min_i w_i$.

Theorem 2. *Suppose that* $\mathbf{w}$ *be a sequence satisfying*

$$0 < q_0 \leq m^2\rho \leq M^2\rho \leq p_0 < 1,$$

where p_0 *and* q_0 *are fixed real numbers in* $(0,1)$*. Then for all* $\varepsilon \in (0,1)$ *with probability at least* $1 - \exp(-\Theta(n^{\varepsilon}))$*,* $G \in G(\mathbf{w})$ *satisfies*

$$c(G) \geq (1-\varepsilon) \log_{\frac{1}{1-p_0}} n.$$

One interpretation of Theorem 2 is that as the network order doubles, on average $\Theta(1)$ more cops are needed to guard the network. For the proof of Theorem 2, we use the following lemma.

Lemma 1. *Let* $0 < p < 1$*,* $r > 0$*, and* $\varepsilon \in (0,1)$ *be fixed. If*

$$d = \left(\log \frac{1}{1-p}\right)^{-1} (1-\varepsilon),$$

then

$$n^{\lfloor d \log n \rfloor + 1} \left(1 - r(1-p)^{\lfloor d \log n \rfloor}\right)^{n - \lfloor d \log n \rfloor - 1} \leq \exp(-\Theta(n^{\varepsilon})). \qquad (1)$$

Proof. It is enough to prove that

$$n^{d \log n + 1} \left(1 - r(1-p)^{d \log n}\right)^{n - d \log n - 1} \leq \exp(-\Theta(n^{\varepsilon})).$$

Now

$$\begin{aligned} n^{d \log n + 1}(1 - r(1-p)^{d \log n})^{n - d \log n - 1} &= n^{d \log n + 1}\left(1 - \frac{r}{n^{1-\varepsilon}}\right)^{n - d \log n - 1} \\ &= \exp\left(f(n)\right), \end{aligned}$$

where

$$f(n) = (d \log n + 1)\log n + (n - d \log n - 1)\log\left(1 - \frac{r}{n^{1-\varepsilon}}\right).$$

However, $\exp\left(f(n)\right) \leq \exp(-\Theta(n^{\varepsilon}))$.

Proof of Theorem 2. We employ the following adjacency property. For a fixed $k > 0$ an integer, we say that G is $(1, k)$-*e.c.* if for each k-set S of vertices of G and vertex $u \notin S$, there is a vertex $z \notin S$ not joined to a vertex in S and joined to u. It is easy to see that if G is $(1, k)$-e.c., then $c(G) \geq k$ (the robber may use the property to escape to a vertex not joined to any vertex occupied by a cop). Let $k = \left\lfloor (1-\varepsilon) \log_{\frac{1}{1-p_0}} n \right\rfloor$. For any graph $G \in G(\mathbf{w})$ we claim that a.a.s. G is $(1, k)$-e.c. Once this is proved, the desired lower bound for the cop number will follow.

Fix S a k-subset of vertices of G and a vertex u not in S. For a vertex

$$x \in U = V(G) \backslash (S \cup \{u\}),$$

the probability that a vertex x is joined to u and to no vertex of S is

$$p_{xu} \prod_{v \in S} (1 - p_{xv}) .$$

Since for $x, y \in U, x \neq y$ and for $v \in S$, the edges xv are chosen independently of the edges yv, the probability that no suitable vertex can be found for this particular S and u is

$$\prod_{x \in U} \left(1 - p_{xu} \prod_{v \in S} (1 - p_{xv}) \right) \leq \left(1 - p'(1-p_0)^k\right)^{n-k-1},$$

where

$$p' = \min_{x \in U} p_{xu}.$$

By hypothesis, $p' \geq q_0 > 0$.

The probability that there exists S and u for which no suitable x can be found is at most

$$n^{k+1}(1 - q_0(1-p_0)^k)^{n-k-1}.$$

By Lemma 1 with $q_0 = r$, $p_0 = p$, we have that

$$n^{k+1}(1 - q_0(1-p_0)^k)^{n-k-1} \leq \exp(-\Theta(n^{\varepsilon})),$$

and the theorem follows. □

In general power law graphs, there may exist an abundance of isolated vertices, even as much as $\Theta(n)$ many. Since the cop number is bounded from below by the number of isolated vertices, we expect the cop number of $G(\mathbf{w})$ to be around cn, for a constant $c \in (0,1)$. We show rigorously that this is indeed the case for random power law graphs, which we now introduce.

Given $\beta > 2$, $d > 0$, and a function $M = M(n)$ (with M tending to infinity with n), we consider the random graph with given expected degrees $w_i > 0$, where

$$w_i = ci^{-\frac{1}{\beta-1}} \tag{2}$$

for i satisfying $i_0 \leq i < n + i_0$. The term c depends on β and d, and i_0 depends also on M; namely,

$$c = \left(\frac{\beta-2}{\beta-1}\right) dn^{\frac{1}{\beta-1}}, \qquad i_0 = n\left(\frac{d}{M}\left(\frac{\beta-2}{\beta-1}\right)\right)^{\beta-1}. \tag{3}$$

It is not hard to show (see [5, 6]) that a.a.s. the random graphs with the expected degrees satisfying (2) and (3) follow a power law degree distribution with exponent β, average degree $d(1+o(1))$, and maximum degree $M(1+o(1))$.

We prove the following result for the cop number of a random power law graph, showing the cop number is a.a.s. equal to $\Theta(n)$.

Theorem 3. *For a random power law graph $G(\mathbf{w})$ with exponent $\beta > 2$ and average degree d, for all $\varepsilon > 0$, a.a.s. the following hold.*

1. *If X is the random variable denoting the number of isolated vertices in $G(\mathbf{w})$, then*

$$X \;= (1+o(1))n \int_0^1 \exp\left(-d\frac{\beta-2}{\beta-1}x^{-1/(\beta-1)}\right) dx.$$

2. *For $a \in (0,1)$, define*

$$f(a) = a + \int_a^1 \exp\left(-d\frac{\beta-2}{\beta-1}a^{(\beta-2)/(\beta-1)}x^{-1/(\beta-1)}\right) dx.$$

Then

$$c(G) \;\leq\; (1+o(1))n \min_{0<a<1} f(a).$$

The theorem demonstrates that the cop number of random power law graphs is a.a.s. $\Theta(n)$, and so is of much larger order than the logarithmic cop number of $G(n,p)$ random graphs. Hence, we should expect in real-world power law graphs such as the web graph that the cop number of order is large, and it would be interesting to conduct experiments which corroborate this claim.

The integrals in the statement of Theorem 3 do not possess closed-form solutions in general. For the integral in item 1, we have that

$$\int_0^1 \exp\left(-tx^{-1/(\beta-1)}\right) dx = e^{-t}\sum_{j=0}^{\infty} \frac{\Gamma(2-\beta)}{\Gamma(2-\beta+j)} t^j + \frac{\pi \csc(\pi\beta) t^{\beta-1}}{\Gamma(\beta-1)},$$

where $t = \frac{d(\beta-2)}{\beta-1}$. The integral in item 2 may be evaluated in cases depending on β. For example, if $2 < \beta < 3$, then the integral

$$\int_a^1 \exp\left(-t\alpha_2 x^{-1/(\beta-1)}\right) dx$$

equals

$$e^{-\alpha_2 t}\sum_{j=0}^{\infty} \frac{\Gamma(2-\beta)}{\Gamma(2-\beta+j)} t^j {\alpha_2}^j - e^{-\alpha_1 t}\sum_{j=0}^{\infty} \frac{\Gamma(2-\beta)}{\Gamma(2-\beta+j)} t^j {\alpha_1}^j + \frac{\pi \csc(\pi\beta) t^{\beta-1}}{\Gamma(\beta-1)} t^{\beta-1} a^{\beta-3}(a-1),$$

where $\alpha_1 = a^{(\beta-3)/(\beta-1)}$ and $\alpha_2 = a^{(b-2)/(b-1)}$.

We supply numerical values for lower/upper bounds of the cop number of $G(\mathbf{w})$ when $d = 10, 20$ and $\beta = 2.1, 2.7$.

	10	20
2.1	$0.1806/0.2940$	$0.5112 \cdot 10^{-1}/0.1265$
2.7	$0.4270 \cdot 10^{-2}/0.1895$	$0.4205 \cdot 10^{-4}/0.8261 \cdot 10^{-1}$

The proof of Theorem 3 requires some background on the domination number of a graph. A set of vertices S is a *dominating set* in

G if each vertex not in S is joined to some vertex of S. The *domination number* of G, written $\gamma(G)$, is the minimum cardinality of a dominating set in G. An easy observation is that

$$c(G) \leq \gamma(G), \tag{4}$$

(place a cop on each vertex of dominating set with minimum cardinality). However, if $n \geq 2$, then $c(P_n) = 1$ (where P_n is a path with n vertices) and $\gamma(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$. The bound of (4) while useful, is far from tight in general. Domination in models for self-organizing networks were considered in Cooper et al. [7].

Proof of Theorem 3. The probability that the vertex i for $i_0 \leq i < n + i_0$ (that is, the vertex i corresponds to the weight w_i) is isolated is equal to

$$\begin{aligned} p_i &= \prod_{j, j \neq i} (1 - w_i w_j \rho) \\ &= \prod_{j, j \neq i} \exp\left(-(1 + o(1)) w_i w_j \rho\right) \\ &= \exp\left(-(1 + o(1)) w_i \rho \sum_{j, j \neq i} w_j\right) \\ &= \exp\left(-(1 + o(1)) w_i\right) . \end{aligned} \tag{5}$$

Let X_i be an indicator random variable for the event that the vertex i is isolated. Then

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p_i$$

for $i_0 \leq i < n + i_0$.

As $X = \sum_{i_0 \leq i < n + i_0} X_i$, it follows from (5) that the expected value of X is

$$\begin{aligned} \sum_{i_0 \leq i < n + i_0} p_i &= (1 + o(1)) n \int_0^1 \exp\left(-(1 + o(1)) c (xn)^{-1/(\beta - 1)}\right) dx \\ &= (1 + o(1)) n \int_0^1 \exp\left(-d \frac{\beta - 2}{\beta - 1} x^{-1/(\beta - 1)}\right) dx . \end{aligned}$$

A sum of independent random variables with large enough expected value is not too far from its mean (see, for example, Theorem 2.8

in [8]). Thus, the number of isolated vertices in $G(\mathbf{w})$ is a.a.s. equal to

$$X = (1+o(1))n \int_0^1 \exp\left(-d\frac{\beta-2}{\beta-1}x^{-1/(\beta-1)}\right) dx\,.$$

Item (1) now follows.

For item (2), we apply (4). Consider $A \subset V$ of first $\lfloor an \rfloor$ vertices from $i_0, \ldots, n+i_0$. Let $B \subset V \setminus A$ denote the set of vertices that do not have a neighbour in A. Then $D = A \cup B$ is a dominating set, and we now estimate the cardinality of D.

Consider the vertex i, $an < i < n+i_0$. Since $i_0 = o(n)$, there is $b \in (0,1]$ such that $i = (1+o(1))bn$. The probability that i does not have a neighbour in A is equal to

$$\begin{aligned}
q_i &= \prod_{j<an+i_0} (1 - w_i w_j \rho) \\
&= \exp\left(-(1+o(1))w_i\rho \sum_{j<an+i_0} w_j\right) \\
&= \exp\left(-(1+o(1))c(bn)^{-1/(\beta-1)}(dn)^{-1}n \int_0^a c(xn)^{-1/(\beta-1)}dx\right) \\
&= \exp\left(-(1+o(1))d\left(\frac{\beta-2}{\beta-1}\right)^2 b^{-1/(\beta-1)} \int_0^a x^{-1/(\beta-1)}dx\right) \\
&= (1+o(1))\exp\left(-d\frac{\beta-2}{\beta-1}b^{-1/(\beta-1)}a^{(\beta-2)/(\beta-1)}\right)\,.
\end{aligned}$$

Thus, using Chernoff's bound, we obtain that a.a.s.

$$|B| = (1+o(1))n \int_a^1 \exp\left(-d\frac{\beta-2}{\beta-1}a^{(\beta-2)/(\beta-1)}x^{-1/(\beta-1)}\right) dx\,,$$

and that a.a.s.

$$|D| = |A \cup B| = an + (1+o(1))n \int_a^1 \exp\left(-d\frac{\beta-2}{\beta-1}a^{(\beta-2)/(\beta-1)}x^{-1/(\beta-1)}\right) dx\,.$$

Item (2) follows as the above estimate of $|D|$ holds for every $a \in (0,1)$. □

As the number of isolated nodes is a lower bound for the domination number of a graph, the proof of Theorem 3 shows that a.a.s. the domination number of random power law graphs is $\Theta(n)$. An analogous result was found in [7] for graphs generated by the preferential attachment model.

3 The cop number in $G(n, p)$ random graphs

The cop number of random graphs $G(n,p)$ for a constant $p \in (0,1)$ was first studied in [3], who proved Theorem 1. We now consider the cop number of $G(n, p(n))$ when $p(n)$ is a function of n. We will abuse notation and refer to p rather than $p(n)$.

Wieland and Godbole [11] proved the following two-point concentration for the domination number of random graphs $G(n,p)$ for p approaching zero sufficiently slowly as $n \to \infty$. Let $\mathbb{L}n = \log_{\frac{1}{1-p}} n$, and define

$$f(p,n) = \lfloor \mathbb{L}n - \mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 2.$$

Theorem 4. *Let p_0 be the smallest p for which*

$$p^2/40 \geq [\log((\log^2 n)/p)]/\log n \tag{6}$$

holds. A.a.s. $G \in G(n,p)$ and $p \geq p_0(n)$ satisfies

$$f(p,n) - 1 \leq \gamma(G) \leq f(p,n).$$

In particular,

$$\gamma(G) = f(p,n)(1 + o(1)).$$

We obtain a concentration result for the cop number of the random graphs $G(n,p)$ where p satisfies (6). Define

$$g(p,n) = \lfloor \mathbb{L}n - 2\mathbb{L}((\mathbb{L}n)(\log n)) \rfloor + 1.$$

Note that $g(p,n) \leq f(p,n)$, and $g(p,n) = f(p,n)(1 + o(1))$.

Theorem 5. *For $G \in G(n,p)$ and $p \geq p_0$, where p_0 is the smallest p for which (6) holds, a.a.s.*

$$g(p,n) \leq c(G) \leq f(p,n).$$

In particular,

$$c(G) = f(s,n)(1 + o(1)).$$

The proof will follow from Theorem 4 if we can establish the lower bound for cop number of $G(n,p)$. We need the following lemma.

Lemma 2. *Let* $k = \mathbb{L}n - 2\mathbb{L}((\mathbb{L}n)(\log n))$. *If*

$$p \geq d\log^2 n/\sqrt{n} \tag{7}$$

where $d > 1$ *is a fixed constant not depending on* n, *then*

$$\lim_{n\to\infty} (k+1)\log n + (n-k-1)\log(1-p(1-p)^k) = -\infty. \tag{8}$$

Proof. By an elementary but tedious analysis we have by (7) that

$$(n-k-1)\log(1-p)\log\left(1-p(1-p)^k\right) = \Omega\left(\log^4 n\right), \tag{9}$$

and

$$-(k+1)\log(1-p)\log n = O\left(\log^2 n\right). \tag{10}$$

By (9) and (10), we obtain that

$$\begin{aligned}
&\lim_{n\to\infty} ((k+1)\log n + (n-k-1)\log(1-p(1-p)^k)) \\
&= \lim_{n\to\infty} \frac{(k+1)\log(1-p)\log n + (n-k-1)\log(1-p)\log(1-p(1-p)^k)}{\log(1-p)} \\
&= -\infty,
\end{aligned}$$

as desired.

Proof of Theorem 5. Let $k = \mathbb{L}n - 2\mathbb{L}((\mathbb{L}n)(\log n))$. Note that the probability that G is not $(1, \lfloor k \rfloor)$-e.c. is at most

$$f(n,k,p) = n^{\lfloor k \rfloor + 1}(1-p(1-p)^{\lfloor k \rfloor})^{n-\lfloor k \rfloor - 1}.$$

To show that

$$n^{\lfloor k \rfloor + 1}(1-p(1-p)^{\lfloor k \rfloor})^{n-\lfloor k \rfloor - 1} = o(1),$$

it suffices to show that

$$n^{k+1}(1-p(1-p)^k)^{n-k-1} = o(1). \tag{11}$$

Note that (6) implies (7). As (11) is equivalent to (8), the result follows by Lemma 2. □

We last consider the cop number of the random graphs $G(n,p)$ for p approaching zero very fast. For example, if $p = o(1/n^2)$, a.a.s. $G \in G(n,p)$ is empty. So in this range of p, a.a.s. the cop number of G is n. We now consider the case when $p = d/n$ for constant $d \in (0,1)$. Bollobás [4] proved the following result.

Theorem 6. *Let $0 < d < 1$, $p = d/n$, and let X be the number of tree connected components of $G(n,p)$. Then the expectation of X is*

$$\mathbb{E}(X) = u(d)n + O(1),$$

where

$$u(d) = \frac{1}{d}\sum_{k=1}^{\infty}\frac{k^{k-2}}{k!}(de^{-d})^k.$$

A.a.s. $G(n,p)$ satisfies

$$|X| = u(d)n(1+o(1)).$$

We note that $u(d) \in (0,1)$. A graph is *unicyclic* if it contains exactly one cycle.

Theorem 7. *Let $0 < d < 1$ and $p = d/n$. Then a.a.s. $G \in G(n,p)$ is such that every connected component is a tree or a unicyclic graph, and there are at most $\log\log n$ vertices in the unicyclic components.*

Trees are cop-win graphs, while unicyclic graphs have cop number at most 2. Each tree component requires exactly one cop, while there are at most $2\log\log n$ many cops needed for all the unicyclic components. Hence, the number of cops on the unicyclic components becomes negligible in contrast to the number of cops on tree components. Therefore, from Theorems 6 and 7 we have the following concentration result.

Corollary 1. *Let $0 < d < 1$, $p = d/n$. Then for the graph $G \in G(n,p)$,*

$$\mathbb{E}(c(G)) = u(d)n + O(\log\log n).$$

A.a.s. $G \in G(n,p)$ satisfies

$$c(G) = u(d)n(1+o(1)).$$

Concentration results for the cop number of $G(n,p)$ with p in other ranges (such as just after the phase transition $p \sim c/n$ with $c > 1$) remain open.

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