Infinite random graphs and properties of metrics

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Abstract We give a survey of recent developments in the theory of countably infinite random geometric graphs. Classical results of Erdős and Rényi establish that countably infinite random graphs are isomorphic with probability 1. Infinite random graphs have vertices identified with points in a metric space, and edges are added with a given probability dependent on the relative location of their endpoints. The probability that infinite random geometric graphs are isomorphic is considered. The metric spaces where such a unique isotype emerges are indeed fairly rare, and specifically arise in the context of finite dimensional normed spaces equipped with the ℓ_∞-metric. We survey negative results for random geometric graphs in the cases of the Euclidean and hexagonal metric. Recent work which considers infinite random geometric graphs in the general setting of normed linear spaces is described. Open problems in the area are provided in the final section.

1 Introduction

Geometric random graph models play an emerging role in the modelling of real-world networks such as on-line social networks [8, 9, 13], wireless networks [21], and the web graph [1, 20]. In such stochastic models, vertices of the network are represented by points in a suitably chosen metric space, and edges are chosen by a mixture of relative proximity of the vertices and probabilistic rules. In real-world networks, the underlying metric space is a representation of the hidden reality that leads to the formation of edges. Such networks can be viewed as embedded in a

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feature space, where vertices with similar features are more closely positioned. An example arises in the social sciences called Blau space [27]. In Blau space, agents in the social network correspond to points in a metric space, and the relative position of nodes follows the principle of *homophily*: nodes with similar socio-demographics are closer together in the space. The web graph may be viewed in *topic space*, where web pages with similar topics are closer to each other. We note that the theory of random geometric graphs has been extensively developed; see, for example, [3, 18, 23, 25, 33], and the books [28, 29]).

While random finite geometric graphs have been closely investigated, the same is not true for the infinite case. One of the most studied examples of an infinite limit graph arising from a stochastic model is the infinite random graph. The probability space $G(\mathbb{N}, p)$ consists of graphs with vertices $\mathbb{N}$, so that each distinct pair of integers is adjacent independently with a fixed probability $p \in (0,1)$. Erdős and Rényi [19] discovered that with probability 1, all $G \in G(\mathbb{N}, p)$ are isomorphic. A graph G is *existentially closed* (or *e.c.*) if for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ adjacent to all vertices of A and to no vertex of B. We say that z is *correctly joined* to A and B. The unique isomorphism type of countably infinite e.c. graph is named the *infinite random graph*, or the *Rado* graph, and is written R. The graph R possesses many properties such as homogeneity, universality, and inexhaustibility. See Chapter 6 of [6] and the surveys [15, 16] for additional background on R.

We considered infinite random geometric graphs first in [10], and studied in that paper the isomorphism types of their associated countable limit structures. The following stochastic model introduced in [10] is sufficiently general for our purposes. Consider a metric space V with distance function

$$d : V \times V \to \mathbb{R},$$

a positive real number δ, a countable subset S of V, and $p \in (0,1)$. The *Local Area Random Graph* $\mathrm{LARG}(V, \delta, p)$ has vertices V, and for each pair of vertices u and v with $d(u,v) < \delta$, an edge is added independently with probability p. In other words, we consider a random, constant-radius disk model on a subset of a metric space. Note that V may be either finite or infinite, although we will consider only V infinite in this chapter.

The LARG model generalizes well-known classes of random graphs. For example, special cases of the LARG model include the random geometric graphs (where $p = 1$), and the binomial random graph $G(n, p)$ (where S has finite diameter D, and $\delta \geq D$).

Our focus will be on the case of separable metric spaces (that is, those containing a countable dense set), where S is chosen as a countable dense subset. Such metric spaces are precisely the second-countable ones (that is, containing a countable base). Following notation introduced in [4], we say that a countable dense set S is *Rado* if the resulting graph is with probability 1 unique up to isomorphism, for any $p \in (0,1)$, and we say it is *strongly non-Rado* if any two such graphs are with probability 1 not isomorphic. Note that these are properties of subsets of metric spaces. A fundamental question when studying graphs generated by the LARG model is to determine which sets are Rado or strongly non-Rado. Perhaps surprisingly, as described in Section 4, there are sets S which are neither Rado nor strongly non-Rado.

In this chapter, we survey results (with proofs omitted or sketched) on the Rado property of dense subsets of normed spaces such as finite dimensional ℓ_p spaces. A fundamental tool in the analysis of infinite geometric graphs is the geometric e.c. property, discussed in Section 2. We then describe an infinite family of Rado sets in Section 3, working in the normed spaces ℓ_∞^d of dimension d. Strongly non-Rado sets are described in ℓ_2^2 and under the hexagonal metric in the plane in Section 4. The recent results of [12] and [4] are then surveyed in Section 5, which shows that almost all sets are non-Rado in a normed space, unless the space is isomorphic to ℓ_∞^d for some d. Interestingly, the tools used in [4] come from classical results in functional analysis. We finish with a set of open problems.

Throughout, all graphs considered are simple, undirected, and countable unless otherwise stated. The cardinality of the natural numbers is denoted by $\aleph_0$. We will encounter two distinct notions of distance: metric distance and graph distance. In a metric space with metric d, we write $d(u,v)$ for the metric distance of the points. For a graph G, we write $d_G(u,v)$ for the graph distance. For a real number $1 \leq p \leq \infty$ and $d \geq 1$ an integer, the vector space $\mathbb{R}^d$ of dimension d equipped with the metric derived from the p-norm is denoted by ℓ_p^d. Given a metric space S with distance function d, denote the (open) *ball of radius* δ *around* x by

$$B_\delta(x) = \{u \in S : d(u,x) < \delta\}.$$

We will sometimes just refer to $B_\delta(x)$ as a δ-*ball* or *ball of radius* δ. A subset V is *dense* in S if for every point $x \in S$, every ball around x contains at least one point from V. We refer to $u \in S$ as points or vertices, depending on the context. For a reference on graph theory the reader is directed to [17, 34], while [14] is a reference on metric spaces.

2 Geometrically e.c. graphs

Adjacency properties have an important role in characterizing infinite random geometric graphs. For a survey of adjacency properties, see [7]. A graph G is *existentially closed* or *e.c.* if for all finite sets A and B of disjoint vertices, there is a vertex $z \notin A \cup B$ that is *correctly joined* to A and B; that is, z is joined to each vertex in A and to no vertex in B. Any two countable e.c. graphs are isomorphic; the isomorphism type is named the *infinite random* or *Rado graph*, and is written R. A central result in infinite graph theory was proven by Erdős and Rényi [19] which states that with probability 1, a countably infinite random graph is isomorphic to R.

We next consider a geometric analogue of the e.c. property. Let $G = (V,E)$ be a graph whose vertices are points in the metric space S with metric d. The graph G

is *geometrically e.c. at level* δ (or δ*-g.e.c.*) if for all δ' so that $0 < \delta' < \delta$, for all $x \in V$, and for all disjoint finite sets A and B so that $A \cup B \in B_\delta(x)$, there exists a vertex $z \notin A \cup B \cup \{x\}$ so that

(i) z is correctly joined to A and B,
(ii) for all $u \in A \cup B$, $d(u,z) < \delta$, and
(iii) $d(x,z) < \delta'$.

This definition implies that V is dense in itself, since we may choose A and B to be empty. Further, if G is δ-g.e.c., then G is δ'-g.e.c. for any $\delta' < \delta$.

The geometrically e.c. property bears clear similarities with the e.c. property defined in the introduction. The important differences are that a correctly joined vertex must exist only for sets A and B which are contained in an open ball with radius δ and centre x, and it must be possible to choose the vertex z correctly joined to A and B arbitrarily close to x; see Figure 1.

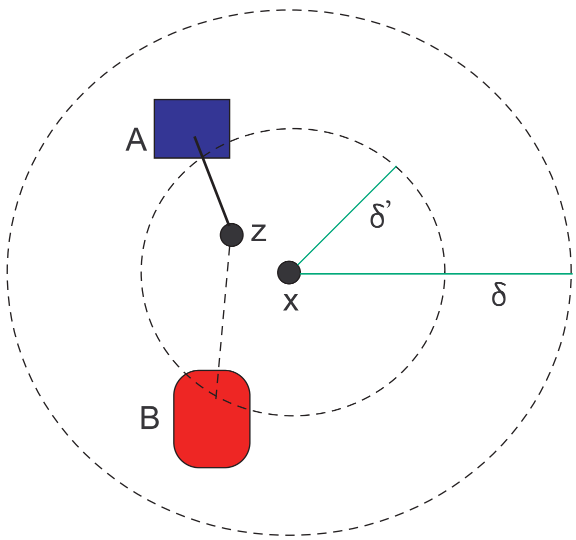

Fig. 1 The δ-g.e.c. property.

The following result demonstrates that graphs generated by the LARG model are typically g.e.c.

Theorem 1 ([10]). *Let* (V,d) *be a metric space and* S *a countable dense subset of* V. *If* $\delta > 0$ *and* $p \in (0,1)$, *then with probability* 1, $\text{LARG}(V,\delta,p)$ *is* δ*-g.e.c.*

A graph $G = (V,E)$ whose vertices are points in the metric space (S,d) has *threshold* δ if for all edges $uv \in E$, $d(u,v) < \delta$. A graph that is geometrically g.e.c. at level δ and has threshold δ is called a *geometric δ-graph*. By definition, a graph G generated by LARG(V,δ,p) has threshold δ, and, if V is countable and dense in itself, G is a geometric δ-graph. Thus, this random graph model generates geometric δ-graphs.

Balls with radius δ in δ-g.e.c. graphs contain copies of R, and hence, contain isomorphic copies of all countable graphs.

Theorem 2 ([10]). *Let $U \subseteq S$ be so that $U \subseteq B_\delta(x)$ for some $x \in U$. Then a δ-g.e.c. graph with vertex set U is e.c., and so is isomorphic to R.*

The following important theorem demonstrates that there exists a close relationship between graph distance and metric distance in any graph that is a geometric δ-graph. We denote the closure of set V in S by $\overline{V}$. The set W is *convex* if for every pair of points x and y in W, there exists a point z such that

$$d(x,z)+d(z,y)=d(x,z).$$

Theorem 3 ([10]). *Let $G=(V,E)$ be geometric δ-graph, and let $\overline{V}$ be convex. Let $u,v \in V$ so that $d(u,v)>\delta$. Then we have that*

$$d_G(u,v)=\lfloor d(u,v)/\delta \rfloor + 1.$$

Theorem 3 directly leads to the following corollary, which is an important tool for proofs in the next two sections.

Corollary 1 ([10]). *If $\overline{V}$ and $\overline{W}$ are convex, and there is a geometric δ-graph with vertices V and a geometric γ-graph graph with vertex set W which are isomorphic via f, then for every pair of vertices $u,v \in V$,*

$$\lfloor d(u,v)/\delta \rfloor = \lfloor d(f(u),f(v))/\gamma \rfloor.$$

Corollary 1 will prove useful as we survey results in the chapter, and it suggests the following generalization of isometry. Given metric spaces (S,d_S) and (T,d_T), sets $V \subseteq S$ and $W \subseteq T$, and positive real numbers δ and γ, a *step-isometry at level* (δ,γ) from V to W is a bijective map $f: V \to W$ with the property that for every pair of vertices $u,v \in V$,

$$\lfloor d_S(u,v)/\delta \rfloor = \lfloor d_T(f(u),f(v))/\gamma \rfloor.$$

Every isometry is a step-isometry, but the converse is false, in general. For example, consider $\mathbb{R}$ with the Euclidean metric, and let $\delta=\gamma=1$. Then $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=\lfloor x \rfloor + (x-\lfloor x \rfloor)^2$ is a step-isometry, but is not an isometry.

Throughout the rest of the chapter, for simplicity and without loss of generality, we assume that $\delta=1$.

3 Isomorphism results for ℓ_∞

The first success in the theory of infinite random geometric graphs came from the realization that for certain dense sets, the LARG model generates graphs that are, with probability 1, isomorphic, regardless of the choice of threshold δ or link probability p. The metric spaces containing these dense sets are the spaces ℓ_∞^d. In fact, we will see that for each d, there is a unique isomorphism type that arises for "almost all" dense vertex sets in $\mathbb{R}^d$.

3.1 ℓ_∞^d gives unique random infinite graphs.

First, we consider ℓ_∞^1. The vertex set V must be dense in $\mathbb{R}$, and it must satisfy the property given by the following definition. A set $V \subseteq \mathbb{R}$ is *integer distance free* (or *idf*) if for any two distinct elements u,v in V, we have that $d(u,v) \notin \mathbb{Z}$.

Theorem 4 ([10]). *Let V and W be two countable dense idf subsets of $\mathbb{R}$. If G is a geometric 1-graph with vertex set V and H is a geometric 1-graph with vertex set W, then $G \cong H$.*

A proof sketch shows how an isomorphism can be constructed. For each $x \in \mathbb{R}$, let $q(x) = \lfloor x \rfloor$ and $r(x) = x - q(x)$. We refer to $r(x)$ is called the *representative* of x and to $q(x)$ as the *quotient*. The following lemma gives an alternative definition of step isometries in terms of quotients and representatives.

Lemma 1. *Let V and W be subsets of $\mathbb{R}$. Assume V and W are idf. Then a bijective function $f : V \to W$ is a step-isometry if and only if the following two conditions hold.*

1. *For every $u,v \in V$, if $r(u) < r(v)$ then $r(f(u)) < r(f(v))$.*
2. *For every $u \in V$, $q(u) = q(f(u))$.*

The proof of Theorem 4 follows using a variant of the back-and-forth method (used to show that R is the unique isotype of e.c. graph). Let $V = \{v_i : i \geq 0\}$ and $W = \{w_i : i \geq 0\}$ and assume that $v_0 = w_0 = 0$. We inductively construct an isomorphism F between pairs of finite sets (V_i, W_i) where V_i contains v_i and W_i contains w_i.

As an additional induction hypothesis we maintain conditions (1) and (2) from Lemma 1.

We first set $f(0) = 0$ and let $V_0 = W_0 = \{0\}$. For the induction step, fix $i \geq 0$. To extend f, we find an image of v_{i+1} which maintains the conditions. Let $v = v_{i+1}$.

Define

$$a = \max\{r(f(u)) : u \in V_i \text{ and } r(u) < r(v)\} \cup \{0\},$$
$$b = \min\{r(f(u)) : u \in V_i \text{ and } r(u) > r(v)\} \cup \{1\}.$$

Since the sets defining a and b are disjoint, and the idf property guarantees that all remainders in W are distinct, we have that $a < b$.

To maintain the induction hypothesis, $r(f(v))$ must lie in $[a,b)$, and $q(f(v))$ must equal to $q(v)$. Set

$$I = (q(v)+a, q(v)+b).$$

Any vertex in $W \cap I$ will qualify as a candidate for $f(v)$, so that conditions (1) and (2) are maintained, and thus, f remains a step-isometry. We must then find a vertex in I that will also guarantee that f is an isomorphism, by making sure it has the correct neighbours. This can be easily done by invoking the g.e.c condition.

Theorem 4 has the following corollaries.

Corollary 2 ([10]). *A countable dense idf set in* $\mathbb{R}$ *is Rado.*

The condition that a dense set should be idf is a weak one. In fact, if we select a dense set V from $\mathbb{R}$ randomly under a reasonable model, then V will be idf with probability 1. Examples of reasonable models to select a dense set V are to form V by taking countably many samples of a distribution on $\mathbb{R}$ whose density function is strictly positive almost everywhere, or to take the union of countably many Poisson point processes [32].

We obtain similar unique isomorphism types of graphs in all dimensions, as proven in the following result. For higher dimensions, we need to extend the definition of idf. Given a set $V \subseteq \mathbb{R}^n$, denote the *i*-th component set of V as:

$$V_i = \{x_i : x \in V\}.$$

A set $V \subseteq \mathbb{R}^d$ is *idf* if the coordinate sets $V_1, \ldots, V_d$ are all idf.

Theorem 5. *Consider the metric space* ℓ_∞^d*. Let* V *and* W *be two countable dense idf sets in* $\mathbb{R}^d$*. If* G *is a geometric* 1*-graph with vertex set* V *and* H *is a geometric* 1*-graph with vertex set* W*, then* $G \cong H$*. In particular, for all choices of dense idf vertex set* V*, there is a unique isomorphism type of geometric g.e.c. graphs in* ℓ_∞^d*, written* GR_d*.*

We inductively construct an isomorphism in a similar way as was sketched for Theorem 4.

For higher dimensions, we do not have a complete characterization of step isometries as in Lemma 1, but we can use this lemma to obtain necessary conditions. Precisely, a bijective function $f : V \to W$ is a step-isometry if the following two conditions hold for all $u, v \in V$ and for all i, $1 \le i \le d$:

$$r(u_i) < r(v_i) \text{ if and only if } r(f(u)_i) < r(f(v)_i),$$
$$q(u_i) = q(f(u)_i).$$

To prove the theorem, we construct an isomorphism between sets V_i and W_i much as in the one-dimensional case. We now sketch how to extend an isomorphism $f : V_i \to W_i$ to a new vertex $v = v_{i+1}$.

For all j, $1 \leq j \leq d$, define

$$a_j = \max\{r(f(u)_j) : u \in V_i \text{ and } r(u_j) \leq r(v_j)\},$$
$$b_j = \min\{r(f(u)_j) : u \in V_i \text{ and } r(u_j) > r(v_j)\}.$$

Note that $a_j < b_j$ for all j. Namely, if not there must exist $1 \leq j \leq d$ and two points $u, w \in W_i$ so that $r(u_j) = r(w_j)$. This contradicts the fact that W is idf.

In order to maintain the induction hypothesis, for all j, $r(f(v))_j$ should lie in interval $[a_j, b_j)$, and $q(f(v)_j)$ should be equal to $q(v_j)$. Let $k_j = q(v_j)$, and consider the product set

$$I = \prod_{1 \leq j \leq d} (q(v_j) + a_j, q(v_j) + b_j).$$

Any vertex in I will qualify as a candidate for $f(v)$ so that f satisfies conditions 1. To complete the proof, we use the fact that W is 1-g.e.c. to show that I contains a vertex that is correctly joined to the vertices in W_i so that f remains an isomorphism.

Corollary 3 ([10]). *For each dimension d, there exists a unique isotype of graph, written GR_d, such that for all countable dense subsets V of $\mathbb{R}$, so that V is idf, for all $p \in (0,1)$, and for all $\delta > 0$, the graph $\mathrm{LARG}(V, \delta, p)$ is isomorphic to GR_d.*

We name GR_d the *infinite random geometric graph of dimension d*. Note that GR_d has infinite diameter for all $d \geq 1$ (unlike R, which has diameter 2). We will explore the structure of the graphs GR_d in the next subsection.

We can apply Theorem 5 to obtain a result about isomorphisms between graphs with vertex sets in $\mathbb{R}^n$ if there exist a special type of map between the sets.

Theorem 6 ([10]). *Consider the metric space ℓ_∞^d. Let V and W be two countable idf sets in $\mathbb{R}^d$. Assume that for all $1 \leq i \leq d$, there exists a step-isometry from V_i to W_i. If G is a geometric 1-graph with vertex set V and H is a geometric 1-graph with vertex set W, then $G \cong H$.*

Note that the statement of this theorem does not require V and W to be dense in $\mathbb{R}^n$, but only in a compact subset of $\mathbb{R}^n$. It is not hard to see, using Lemma 1, that there exists a step-isometry between two intervals (a,b) and (a', b') if their lengths $t_1 = b - a$ and $t_2 = b' - a$ are not integers, and if $\lfloor t_1 \rfloor = \lfloor t_2 \rfloor$. Applying this fact to each dimension, and using the above theorem, we obtain the following corollary.

Corollary 4. *Let V and W be two countable sets in $\mathbb{R}^d$. Suppose that for each $1 \leq i \leq d$, the closures $\overline{V_i}$ and $\overline{W_i}$ are intervals of length $t_{V,i}$, $t_{W,i}$, respectively, where $t_{V,i}$ and $t_{W,i}$ are not integers, and $\lfloor t_{V,i} \rfloor = \lfloor t_{W,i} \rfloor$. Then there exists a step-isometry between V and W. Moreover, if G is a geometric 1-graph with vertex set V and H is a geometric 1-graph with vertex set W, then $G \cong H$.*

Thus, for any d-tuple $(t_1, t_2, \ldots, t_d)$ there is a unique isomorphism type of geometrically e.c. graphs in ℓ_∞^d with countable vertex set, corresponding to 1-g.e.c. graphs whose vertex sets are dense in a set $I_1 \times \ldots \times I_d$, where each I_i is an interval of length $t_i + \varepsilon_i$, and $0 < \varepsilon_i < 1$.

3.2 Properties of the unique limit

One question is to determine whether for distinct dimensions d, the infinite random geometric graphs GR_d are non-isomorphic. We settle this question in the affirmative here using a geometric property of metric spaces.

The *equilateral dimension* of a metric space S is the maximum number of equidistant points in S. In [30], it was proven that in a Banach space of dimension d, the equilateral dimension is bounded above by 2^d, and from below by $\min\{4, d+1\}$. The equilateral dimension of ℓ_∞^d equals 2^d (with the lower bound witnessed by the set of binary vectors), while it equals $d+1$ in Euclidean space ℓ_2^d. Interestingly, the equilateral dimension of ℓ_1^d is unknown, and is claimed to be $2d$ in what is referred to as *Kusner's conjecture*; see [24]. Note that the standard basis vectors and their negatives witness the lower bound of $2d$ in the ℓ_1^d case. See [2] for further background on equilateral dimension of the ℓ_p spaces.

We utilize equilateral dimension in our graph theoretic setting via the following definition. For a graph G and positive integer t, define the *equilateral clique number*, written $\omega(G,t)$, to be the supremum of the cardinalities of a set A of vertices so that every pair of distinct vertices in A have graph distance exactly t. We first provide a lower bound on the equilateral clique number for any geometric 1-graph in ℓ_p^d.

Theorem 7. *Let k be the equilateral dimension of ℓ_p^d, where $d \geq 1$ and $1 \leq p \leq \infty$. Then for any geometric 1-graph G with vertex set V dense in ℓ_p^d, and for all $t \geq 2$, we have that $\omega(G,t) \geq k$.*

Proof. Fix $t \geq 2$. Choose a real number $\varepsilon > 0$ so that $5\varepsilon < 1$. Let S be a subset of ℓ_p^d so that $|S| = k$ and every pair of distinct points in S have distance exactly $t + 3\varepsilon$. For each $x \in S$, choose a vertex $v_x \in V$ such that $d(x, v_x) < \varepsilon$. Set $V_S = \{v_x : x \in S\}$. For all $v_x, v_y \in V_S$, we have that

$$t + \varepsilon = d(x,y) - 2\varepsilon < d(v_x, v_y) < d(x,y) + 2\varepsilon = t + 5\varepsilon < t + 1.$$

Hence, we have that $\lfloor d(v_x, v_y) \rfloor = t$, and so by Theorem 3, we have that the graph distance of v_x to v_y equals t. It follows that $\omega(G,t) \geq k$. $\square$

The following theorem proves that the graphs GR_d are non-isomorphic for distinct dimensions d.

Theorem 8. *For all $d \geq 1$ and positive integers $t \geq 3$, $\omega(GR_d, t) = 2^d$.*

Proof. Recall that ℓ_∞^d has equilateral dimension 2^d. It follows from Theorem 7 and the facts that GR_d can be realized as a geometric 1-graph, , that $\omega(GR_d, t) \geq 2^d$.

Now, to prove that $\omega(GR_d, t) \leq 2^d$, we use induction on $d \geq 1$. In fact, we prove a stronger statement that ℓ_∞^d does not contain $2^d + 1$ points whose pairwise distance lies strictly between t and $t+1$. The result then follows by Theorem 3.

The case $d = 1$, is elementary, since one cannot find real numbers x, y, and z with the property that $|x-y|$, $|x-z|$, and $|y-z|$ are all greater than 3, but differ by at most 1. The proof for $d = 1$ now follows by Theorem 3.

Now, fix $d \geq 1$ and assume the statement is true for $d = 1$. Let A' be a set of maximum cardinality in ℓ_∞^d such that for all pairs $a, b \in A'$ we have that $\lfloor d(a,b) \rfloor = t$. For all $a = (a_1, a_2, \ldots, a_d) \in A$, let a' be the projection onto the first d dimensions; that is, $a' = (a_1, a_2, \ldots, a_{d-1})$. Assume, without loss of generality, that the origin is an element of A, and $a_1 \geq 0$ for all $a \in A$.

We set

$$A_1 = \{a \in A : 0 \leq a_d < \frac{t+1}{2}\},$$

and

$$A_2 = \{a \in A : \frac{t+1}{2} \leq a_d < t+1\}.$$

It is evident that A_1 and A_2 partition A. Without loss of generality, assume that $|A_1| \geq |A_2|$, so $|A_1| \geq \frac{|A|}{2}$. For all $a, b \in A_1$, by definition we have that $|a_d - b_d| < \frac{t+1}{2} < t$, and also that $d(a,b) \geq t$.

Therefore, the distance between a and b is not achieved in the last coordinate, so $d(a', b') \geq t$, where $a', b' \in \ell_\infty^{d-1}$. Hence, A_1', the projection of A_1 onto the first $d-1$ coordinates, is a set of vertices in ℓ_∞^{d-1} such that for all $a', b' \in A_1'$, $\lfloor d(a', b') \rfloor = t$. It follows by induction that $|A_1| \leq 2^{d-1}$ and so $|A| \leq 2|A_1| \leq 2^d$. $\square$

We note that it is straightforward to prove that $\omega(GR_d, t) = \aleph_0$, where $t = 1, 2$ (since such graphs contain the infinite random graph as an induced subgraph).

Deleting a point from a dense set S with the idf property in $\mathbb{R}^n$ gives another dense set with the idf property. As a consequence of Corollary 6, we have the following *inexhaustibility* property.

Corollary 5 ([10]). *For all $d > 0$ and vertices x in GR_d, $GR_d - x \cong GR_d$.*

The g.e.c. property may be used to prove a relationship between the graphs GR_d for different d. Suppose V is a dense idf subset of $\mathbb{R}^d$, and G a 1-geometric graph with vertex set V (so $G \cong GR_d$). Let $V^* = \{v \in V : 0 < v_d < 1\}$. Then it is not hard to see that the floor of the distance, and thus, the graph distance, between two vertices is determined by the projection onto the first $d-1$ dimensions. It follows that the subgraph induced by V^* is isomorphic to a 1-geometric graph whose vertex set is the projection of V^* onto the first $d-1$ dimension. By Theorem 5, this graph is isomorphic to GR_{d-1}, and since rounded distances are preserved, so are graph distances. Hence, the embedding of this subgraph into G is isometric. Applying this argument repeatedly, we obtain the following theorem.

Theorem 9. *For $j < k$, GR_k contains GR_j as an isometric induced subgraph.*

A *step-isometric isomorphism* is an isomorphism of graphs that is a step-isometry. The following corollary shows that the graphs GR_n act transitively on step-isometric isomorphic induced subgraphs.

Corollary 6 ([10]). *Let G and H be finite induced subgraphs of GR_d for some positive integer d. A step-isometric isomorphism $f : G \to H$ extends to an automorphism of GR_d.*

4 Non-isomorphism results

The normed space ℓ_∞^d is rather special from the viewpoint of infinite random geometric graphs. This notion will be rigorously explored in the next section. Here, we are content to provide some results demonstrating that for some metric spaces all dense subsets are strongly non-Rado.

4.1 The Euclidean metric in the plane

We first consider a result proved in [7]. The familiar Euclidean metric space in the plane gives examples of strongly non-Rado sets.

Theorem 10 ([10]). *If S is a countable set dense in ℓ_2^2, then S is strongly non-Rado.*

The main tool in proving Theorem 10 is the following lemma. For the proof of Theorem 10, we rely on the following geometric lemma.

Lemma 2 ([10]). *Let V and W be dense subsets of ℓ_2^2. Then every step-isometry from V to W is an isometry.*

The lemma is proved by showing, using geometric configurations, that any small "error" in the distances between the vertices leads inevitably to larger errors in the distances between other pairs of vertices, and by repetition of the construction leads to errors that exceed 1. This then contradicts the properties of a step-isometry.

An intuitive interpretation of how this lemma leads to the non-isomorphism result is the following. Any isometry in ℓ_2^2 is determined by the images of three points in general position. Any isomorphism between two geometric 1-graphs G and H must be a step-isometry and thus, an isometry. Suppose that G and H were generated by the LARG process. We have countably many choices for the images, under an isomorphism for G to H, of the first three vertices. Once these images are fixed, the images of the remaining vertices of G and H are completely determined. Thus, if G and H are to be isomorphic, for each of the $2^{\aleph_0}$ pairs of vertices, G and H must agree on the existence of an edge between that pair.

4.2 Hexagonal metric

The hexagonal metric arises in the study of Voronoi diagrams and period graphs (see [22]) in computational geometry, which in turn have applications to nanotechnology. The hexagonal metric arises as a special case of *convex polygonal (or polygon-offset) distance functions*, where distance is in terms of a scaling of a convex polygon containing the origin; see for example, [5]. A precise definition of this metric is given below. We note that *honeycomb networks* formed by tilings by hexagonal meshes have been studied as, among other things, a model of interconnection networks.

We now formally define the hexagonal metric. Consider the vectors

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}, \text{ and } a_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}.$$

These are the normal vectors to the sides of a regular hexagon, as shown in Figure 2. We will calls the three line directions perpendicular to a_1, a_2, a_3 the *principal directions.*

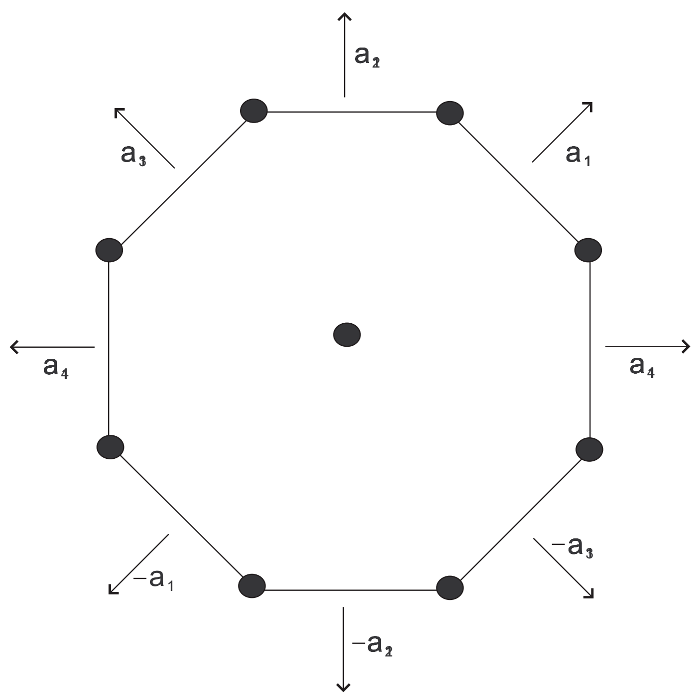

Fig. 2 The vectors a_i.

For $x \in \mathbb{R}^2$ define the *hexagonal norm* of **x** as follows:

$$\|x\|_{\text{hex}} = \max_{i=1,2,3} |a_i \cdot x|,$$

where "$\cdot$" is the dot product of vectors. The *hexagonal metric* in $\mathbb{R}^2$ is derived from the hexagonal norm, and defined by

$$d_{\text{hex}}(x,y) = \|x - y\|_{\text{hex}}.$$

Note that the unit balls with the hexagonal metric are regular hexagons as in Figure 2. We denote the hexagonal metric by d_{hex}.

Theorem 11 ([11]). *Almost all dense sets in $\mathbb{R}^2$ with the d_{hex}-metric are strongly non-Rado.*

The proof Theorem 11 uses the fact that any isomorphism between geometric 1-graphs with vertex set V must be a step-isometry. It is evident that for any point $v \in V$, the balls $B(v,z)$ where $z \in \mathbb{N}$, must be *respected* by any isometry f; that is, any points inside the ball $B(v,z)$ must have an image inside the corresponding ball $B(f(v),z)$. This fact can be used to show that any line with a principal direction and going through a point $v \in V$ must also be respected. Moreover, any line at integer distance from such a line must also be respected. Thus, a finite set of points in V generates a grid of lines, all in principal directions, that must be respected. The regions bounded by this grid are of positive area, so this in itself does not lead to a contradiction. However, it can be shown that any line in a principal direction and going through an intersection point of lines of this grid must also be respected. Applying this fact iteratively leads to the formation of a dense grid of lines which must be respected, derived from only a finite number of points. Thus, any step-isometry is determined by the images of a finite number of points. The non-isomorphism result then follows along the same lines as in the previous case.

5 Normed spaces

In [4], the notions of Rado and strongly Rado sets were introduced, along with general results on the existence and non-existence of Rado sets in the setting of normed spaces. One of the key results of [4] is the following theorem, which establishes the special role of ℓ_∞ in the theory of random geometric graphs. A *random dense set* is a countable union of sets each chosen according to a Poisson point process [32].

Theorem 12 ([4]). *Let S be a finite-dimensional normed space not isometric to ℓ_∞^d. For all $0 < p < 1$, almost every random dense set V is strongly non-Rado.*

An analogue of Theorem 12 was proven in [12], using a different approach (in particular, segmenting the cases into the polygonal and non-polygonal metric cases).

The results of Theorem 12 correspond to a special case of the following key result. As proved in [4], for a finite dimensional normed space S, there exists a unique maximal subspace T isometric to ℓ_∞^d for some d, such that there is a subspace U with $S = U \bigoplus T$ and

$$\|u+w\| = \max\{\|u\|,\|w\|\}$$

for all $u \in U$ and $w \in T$. This is referred to as the ℓ_∞-decomposition of V, and is written $(U \bigoplus \ell_\infty^d)_\infty$.

Theorem 13 ([4]). *Let S be a finite dimensional normed space with ℓ_∞-decomposition $(U \bigoplus \ell_\infty^d)_\infty$, and fix $0 < p < 1$. Then the following hold.*

1. *If $U = 0$ in the ℓ_∞-decomposition, then almost all countable dense sets V are Rado, but there exist countable dense sets which are strongly non-Rado. Further, there exist countable dense sets S for which the probability that two graphs $G, G' \in \mathrm{LARG}(V, 1, p)$ are isomorphic lies strictly between 0 and 1.*
2. *If $S = U$, then all countable dense sets S are strongly non-Rado.*
3. *If $d > 0$ and $U \neq \{0\}$ then almost all countable dense sets V are strongly non-Rado, but there exist countable dense sets V which are Rado. Additionally, there exist countable dense sets for which the probability that two graphs $G, G' \in \mathrm{LARG}(V, 1, p)$ are isomorphic lies strictly between 0 and 1.*

Item (1) in Theorem 13 covers the case where the metric space is ℓ_∞^d, which we discussed in Section 3. Namely, almost all countable dense sets are idf, and thus, the first part of the case follows from Theorem 5 (which proves the stronger result that there is a unique isomorphism type for all countable dense sets that are idf). The remainder of this case can be understood when we consider what happens if V is not idf. In particular, suppose that V contains a subset W of points which have only integer coordinates. Then any isomorphism must be an isometry on W and must fix W. Hence, this restricts the possibilities for an isomorphism. We then have a relationship between the probability that the graphs are isomorphic to the probability that the subgraphs on W are isomorphic. The corresponding probability can be zero, or it can be between 0 and 1, depending on W.

Item (2) is a generalization of the results for ℓ_2^2 and the hexagonal metric discussed above. The proofs rely on the analysis of step-isometries, and their characterization. The proofs employ results from functional analysis such as the Mazur-Ulam theorem, and properties of extreme points in normed spaces. It is shown that any step-isometry f is highly constrained. For example, as part of the proof, it is shown that f generates a lattice derived from the extreme points of the unit ball, and that f must be an isometry on this lattice. The restrictions on step-isometries limit the possibilities for an isomorphism, and the non-isomorphism result then follows by an argument similar to the one sketched in the previous subsections. Item (3) follows similarly. The fact that there exist Rado sets and sets that are neither Rado nor non-Rado follows by construction of special sets where the ℓ_∞ component of the space is somehow dominant for pairs of vertices in the set.

6 Open problems

The most fundamental question in the study of infinite random geometric graphs is to classify Rado sets in all normed spaces. This problem was mentioned in [4], and may be summarized as follows.

Problem 1 [4]: Let V be a normed space with ℓ_∞-decomposition $V = (U \bigoplus \ell_\infty^d)_\infty$ for some $d > 1$. Classify the countable dense Rado sets.

The infinite dimensional case is wide open. The second problem mentioned in [4] is the following.

Problem 2 [4]: Let V be an infinite dimensional normed space. Classify the countable dense Rado sets.

The most concrete example to consider here is the space of bounded sequences, written ℓ_∞^∞, which has dimension $\aleph_0$. A challenge with the space ℓ_∞^∞ from our perspective is that it is not separable. Does such a space even contain Rado sets?

Notice that the LARG model easily adapts to any metric space. Theorem 1, which states that graphs generated with the LARG model are geometrically e.c., holds for any metric space. However, the other results discussed here are all about normed spaces, and the tools developed to prove the results apply only to such spaces. Perhaps the broadest generalization of this line of research is to consider general separable metric spaces.

Problem 3: Let V be a separable metric space. Classify the countable dense Rado sets.

The LARG model extends to the case of uncountable graphs; for example, one may consider the unit disk graph of uncountable dense subsets of the plane equipped with the ℓ_∞ metric. Infinite random geometric graphs where the underlying vertex set is uncountable appear to be more challenging to study, and the classification of such likely relies on various axioms of infinity in set theory. An interesting problem is therefore, the following.

Problem 4: Are there are uncountable Rado sets in normed spaces?

As a final remark, we point out possible connections with model theory. The infinite random graph R is the unique countably graph satisfying almost sure (first-order) theory of graphs (see, for example, [31]). In related work, McColm [26] considered 0-1 laws for *Gilbert* graphs, which in our terminology are finite graphs generated by the LARG model with $p = 1$, where the underlying space is the one-dimensional unit circle S^1 (with metric the distance between reals modulo 2π). As mentioned in [26], it would be interesting to establish whether there exist 0-1 laws for the LARG model in certain metric spaces, and analyze the isomorphism types of countable models of the almost sure theory. In particular, an intriguing question is whether the infinite random geometric graphs GR_d of dimension d play an analogous role to the one played by R in the almost sure theory of geometric graphs.

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