Towards universality in graph bootstrap percolation

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Abstract We report on some topics pursued at the MATRIX event *Combinatorics of McKay and Wormald* regarding graph bootstrap percolation on random graphs. The literature has mostly focussed on balanced template graphs. We are working towards more general results.

Introduction

In graph bootstrap percolation, we start with an Erdős–Rényi random graph $\mathscr{G}_0 = \mathscr{G}_{n,p}$. We fix a *template* graph H to govern the dynamics. Specifically, in step $k \geq 1$ of the process, we obtain $\mathscr{G}_k$ by adding each missing $e \notin E(\mathscr{G}_{k-1})$ that would create a new copy of H. We let $\langle \mathscr{G}_{n,p} \rangle_H = \bigcup_{k \geq 0} \mathscr{G}_k$ denote the closure of these dynamics, and say that $\mathscr{G}_{n,p}$ *H-percolates* if $\langle \mathscr{G}_{n,p} \rangle_H = K_n$. Finally, we let

$$p_c(n,H) = \inf\{p > 0 : \mathbf{P}(\langle \mathscr{G}_{n,p} \rangle_H = K_n) \geq 1/2\}$$

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denote the *critical H-percolation threshold*.

This process was introduced by Balogh, Bollobás and Morris [1], following the early work of Bollobás [5] on *weak saturation*. Indeed, p_c is the point at which the $\mathscr{G}(n,p)$ is likely to be weakly saturated. Further inspiration for this process came from the study of *bootstrap percolation* in statistical physics [6]; see, e.g., Morris [7] for a recent survey. Bootstrap percolation is a simple example of a *cellular automaton*, that is, a process in which the sites in the system change their status depending on the status of their local environment. Interestingly, such local rules can give rise to complex global behavior; see, e.g., the early work of Ulam [8] and von Neumann [9].

In bootstrap percolation, the percolation process is usually started by initially infecting all sites independently with probability p. In the context of graph bootstrap percolation, the sites can be thought of as the edges of the complete graph K_n. A site $e \in E(K_n)$ being initially infected then corresponds to including this edge in our initial graph $\mathscr{G}_0$; indeed, this is precisely how $\mathscr{G}_{n,p}$ is constructed. Hence, from a statistical physics point of view, the study of $p_c(n,H)$ is a natural question, and in some sense greatly generalizes the classical bootstrap percolation model.

From a more combinatorial perspective, let us note that, clearly, K_3-percolation is equivalent to connectivity. Indeed, by induction, it can be seen that the K_3-dynamics turn all paths into cliques. On the other hand, if a graph has two disconnected regions, then the K_3-dynamics will never add an edge from one to the other (consider, towards a contradiction, the first time this happens). Therefore, $p_c(n,K_3)$ is the classical connectivity threshold for $\mathscr{G}_{n,p}$. From this point of view, the study of $p_c(n,H)$ is natural, as these quantities can be seen as thresholds for more general forms of connectivity.

Literature

The best known results are for templates graphs H that are *balanced* in the sense that, for every $e \in E(H)$, the graph $H \setminus e$, obtained from H by removing e, is *2-balanced*. More concretely,

$$\frac{e(F)-1}{v(F)-2} \le \frac{e(H)-2}{v(H)-2},$$

for all proper subgraph F of H with at least 3 edges. Cliques $H = K_r$ are balanced in this sense.

Combining results from [1, 2] it follows that

$$p_c(n,H) = n^{-1/\lambda + o(1)},$$

for all balanced H, where

$$\lambda(H) = \frac{e(H)-2}{v(H)-2}.$$

Problem 1 in [1] asks for $\ell(H)$ such that

$$p_c(n,H) = n^{-\ell(H)+o(1)},$$

for *all* graphs H, and is a major open problem in the area.

In [3] it is observed that the random graph $H = \mathscr{G}(k,1/2)$ is balanced with high probability, as $k \to \infty$. In this sense, $\ell = 1/\lambda$ for *most* graphs H, since $\mathscr{G}(k,1/2)$ is a uniformly random graph on k vertices.

The value of ℓ is only known for a handful of (non-trivial) unbalanced graphs. For instance, $\ell(K_{2,4}) = 10/13$ (see [4]) and $\ell(DD_r) = r/(\binom{r}{2}+1)$ (see [1]), where DD_r is the double dumbbell graph, obtained by adding two disjoint edges between two disjoint copies of K_r.

A general lower bound $p_c \geq n^{-1/\hat{\lambda}+o(1)}$ is proved in [2], for all H with at least four vertices and minimum degree at least two (which covers all non-trivial cases). Here,

$$\hat{\lambda}(H) = \min_F \frac{e(H)-e(F)-1}{v(H)-v(F)},$$

over all subgraphs F with $2 \leq v(F) < v(H)$. It can be shown that $\hat{\lambda} \leq \lambda$ and $\hat{\lambda} = \lambda$ if and only if H is balanced. We note that $\hat{\lambda}$ can be viewed as the edge-per-vertex cost of the most efficient way of adding a new edge via the H-dynamics; see, e.g., Fig. 1 in [3].

Interestingly, $\ell = 1/\hat{\lambda}$ when $H = DD_r$; however, if $H = K_{2,4}$ then $1/\lambda < \ell < 1/\hat{\lambda}$. Indeed, the general behavior of ℓ remains largely mysterious.

A second fascinating question is Problem 2 in [1] that asks for what H is the threshold p_c *sharp*, meaning that the *critical window*, between which $\mathbf{P}(\langle \mathscr{G}_{n,p} \rangle_H = K_n)$ goes from ε to $1-\varepsilon$, has width $o(p_c)$.

Report

While at MATRIX, the authors focussed on Problems 1 and 2 in [1], as discussed above. We have made some progress, which we plan to continue developing together. We aim to determine ℓ for an arbitrary H and to characterize template graphs H that certify sharpness, at least partially.

Question

Many interesting open problems remain. Let us finish this note with one such question.

As discussed at the end of [3], the graph $H = \mathscr{G}(k,\alpha)$ is balanced (with high probability, as $k \to \infty$) provided that $\alpha > \beta_*(\log k)/k$, where $\beta_* = 2/\log(e/2)$. Since little is known about p_c when H is unbalanced, it would be useful to understand at least the typical behavior. That is, it would be interesting to study ℓ for such H as above, as β ranges over $(1, \beta_*)$, so that H is connected but unbalanced.

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