Makespan Trade-offs for Visiting Triangle Edges (Extended Abstract) ⋆

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Abstract. We study a primitive vehicle routing-type problem in which a fleet of n unit speed robots start from a point within a non-obtuse triangle Δ, where $n \in \{1,2,3\}$. The goal is to design robots' trajectories so as to visit all edges of the triangle with the smallest visitation time makespan. We begin our study by introducing a framework for subdividing Δ into regions with respect to the type of optimal trajectory that each point P admits, pertaining to the order that edges are visited and to how the cost of the minimum makespan $R_n(P)$ is determined, for $n \in \{1,2,3\}$. These subdivisions are the starting points for our main result, which is to study makespan trade-offs with respect to the size of the fleet. In particular, we define $\mathscr{R}_{n,m}(\Delta) = \max_{P\in\Delta} R_n(P)/R_m(P)$, and we prove that, over all non-obtuse triangles Δ: (i) $\mathscr{R}_{1,3}(\Delta)$ ranges from $\sqrt{10}$ to 4, (ii) $\mathscr{R}_{2,3}(\Delta)$ ranges from $\sqrt{2}$ to 2, and (iii) $\mathscr{R}_{1,2}(\Delta)$ ranges from 5/2 to 3. In every case, we pinpoint the starting points within every triangle Δ that maximize $\mathscr{R}_{n,m}(\Delta)$, as well as we identify the triangles that determine all $\inf_\Delta \mathscr{R}_{n,m}(\Delta)$ and $\sup_\Delta \mathscr{R}_{n,m}(\Delta)$ over the set of non-obtuse triangles.

Keywords: 2-Dimensional Search and Navigation · Vehicle Routing · Triangle · Make-span · Trade-offs.

1 Introduction

Vehicle routing problems form a decades old paradigm of combinatorial optimization questions. In the simplest form, the input is a fleet of robots (vehicles) with some starting locations, together with stationary targets that need to be visited (served). Feasible solutions are robots' trajectories that eventually visit every target, while the objective is to minimize either the total length of traversed trajectories or the time that the last target is visited.

Vehicle routing problems are typically NP-hard in the number of targets. The case of 1 robot in a discrete topology corresponds to the celebrated Traveling Salesman Problem whose variations are treated in numerous papers and books. Similarly, numerous vehicle routing-type problems have been proposed and studied, varying with respect to the number of robots, the domain's topology and the solutions' specs, among others.

We deviate from all previous approaches and we focus on efficiency trade-offs, with respect to the fleet size, of a seemingly simple geometric variation of a vehicle routing-type problem in which targets are the edges of a non-obtuse triangle. The optimization problem of visiting all these three targets (edges), with either 1, 2 or 3 robots, is computationally degenerate. Indeed, even in the most interesting case of 1 robot, an optimal solution for a given starting point can be found by comparing a small number of candidate optimal trajectories (that can be efficiently constructed geometrically). From a combinatorial geometric perspective, however, the question of characterizing the points of an arbitrary non-obtuse triangle with respect to optimal trajectories they admit when served by 1 or 2 robots, e.g. the order that targets are visited, is far from trivial (and in fact it is still eluding us in its generality).

In the same direction, we ask a more general question: Given an arbitrary non-obtuse triangle, what is the worst-case trade-off ratio of the cost of serving its edges with different number of robots, over

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all starting points? Moreover, what is the smallest and what is the largest such value as we range over all non-obtuse triangles? Our main contributions pertain to the development of a technical geometric framework that allows us to pinpoint exactly the best-case and worst-case non-obtuse triangles, along with the worst-case starting points that are responsible for the extreme values of these trade-off ratios. To the best of our knowledge, the study of efficiency trade-offs with respect to fleet sizes is novel, at least for vehicle routing type problems or even in the realm of combinatorial geometry.

1.1 Motivation & Related Work

Our problem is related to a number of topics including vehicle routing problems, the (geometric) traveling salesman problem, and search and exploration games. Indeed, the main motivation for our problem comes from the so-called shoreline search problem, first introduced in [4]. In this problem, a unit robot is searching for a hidden line on the plane (unlike our problem in which the triangle edges are visible). The objective is to visit the line as fast as possible, relative to the distance of the line to the initial placement of the robot. The best algorithm known for this problem has performance of roughly 13.81, and only very weak (unconditional) lower bounds are known [3]. Only recently, the problem of searching with multiple robots was revisited, and new lower bounds were proven in [1,8].

As it is common in online problems, a typical argument for a lower bound for the shoreline problem lets an arbitrary algorithm perform for a certain time until the hidden item is placed at a location that cannot have been visited before by the robot. The lower bound then is obtained by adding the elapsed time with the distance of the robot to the hidden item (the line), since at this point one may only assume that the (online) algorithm has full knowledge of the input. Applying this strategy to the shoreline problem, one is left with the problem of identifying a number of lines, as close as possible to the starting point of the robot, and then computing the shortest trajectory of the robot that could visit them all, exactly as in our problem. In the simplest configuration that could result strong bounds, one would identify three lines, forming a non-obtuse triangle. The latter is also the motivating reason we restricted our attention to non-obtuse triangles (a second reason has to do with the optimal visitation cost of 3 robots, which for non-obtuse triangles is defined as the maximum distance over all triangle edges, treated as lines).

Our problem could also be classified as a vehicle routing-type problem, the first of which was introduced in [6] more than 60 years ago. The objective in vehicle routing problems is typically to minimize the visitation time (makespan) or the total distance traveled for serving a number of targets given a fleet of (usually capacitated) robots, see surveys [11,14] for early results. Even though the underlying domain is usually discrete, geometric vehicle type problems have been studied extensively too, e.g. in [7]. Over time, the number of proposed vehicle routing variations is so vast that surveys for the problem are commonly subject-focused; see surveys [10,12,13] for three relatively recent examples.

Famously, vehicle routing problems generalize the celebrated traveling salesman problem (TSP) where a number of targets need to be toured efficiently by one vehicle. Similarly to vehicle routing, TSP has seen numerous variations, including geometric [2,9], where in the latter work targets are lines. The natural extension of the problem to multiple vehicles is known as the multiple traveling salesman problem [5], a relaxation to vehicle routing problems where vehicles are un-capacitated. The latter problem has also seen variations where the initial deployment of the vehicles is either from a single location (single depot), as in our problem, or from multiple locations.

1.2 A Note on our Contributions & Paper Organization

We introduce and study a novel concept of efficiency/fleet size trade-offs in a special geometric vehicle routing-type problem that we believe is interesting in its own right. Deviating from the standard combinatorial perspective of the problem, we focus on the seemingly simple case of visiting the three edges of a non-obtuse triangle with $n \in \{1,2,3\}$ robots. Interestingly, the problem of characterizing the starting points within arbitrary non-obtuse triangles with respect to structural properties of the optimal trajectories they admit is a challenging question. More specifically, one would expect that the latter characterization is a prerequisite in order to analyze efficiency trade-offs when serving with different number of robots, over all triangles. Contrary to this intuition, and without fully characterizing the starting points of arbitrary triangles, we develop a framework that allows us (a) to pinpoint the starting points of any triangle at which these (worst-case) trade-offs attain their maximum values, and (b) to identify the extreme cases of non-obtuse triangles that set the boundaries of the inf and sup values of these worst-case trade-offs.

This is an extended abstract of our work. Due to space limitations we only present the backbone of our arguments, along with the critical intermediate lemmata that are invoked toward proving our main results. The full version of the paper with a revised structure is attached as an appendix at the end of the submission. The paper organization of the extended abstract is as follows. In Section 2.1 we give a formal definition of the problem we study, and we quantify our main contributions. Then, in Section 2.2 we establish some of the necessary terminology and we present some preliminary and important observations. The technical analysis starts in Section 3. First, in Section 3.1 we study the simpler problem of visiting only two triangle edges with one robot. It is followed by Section 3.2, were we find optimal trajectories for visiting all three triangle edges by one robot in a predetermined order. That brings us to Section 4 where we characterize triangle regions with respect to the optimal visitation strategies they admit, for 3 (Section 4.1), 2 (Section 4.2) and 1 robots (Section 4.1). Equipped with that machinery, we outline in Section 5 how our main contributions are proved. More specifically, Section 5.1, Section 5.2 and Section 5.3 discuss trade-offs between 1 and 3, 2 and 3, and 1 and 2 robots, respectively. Finally in Section 6 we conclude with some open questions.

2 Our Results & Basic Terminology and Observations

2.1 Problem Definition & Main Contributions

We consider the family of non-obtuse triangles $\mathscr{D}$, equipped with the Euclidean distance. For any $n \in \{1,2,3\}$, any given triangle $\Delta \in \mathscr{D}$, and any point P in the triangle, denoted by $P \in \Delta$, we consider a fleet of n unit speed robots starting at point P. A feasible solution to the triangle Δ visitation problem with n robots starting from P is given by robots' trajectories that eventually visit every edge of Δ, that is, each edge needs to be touched by at least one robot in any of its points including the endpoints. The visitation cost of a feasible solution is defined as the makespan of robots' trajectory lengths, or equivalently as the first time by which every edge is touched by some robot. By $R_n(\Delta, P)$ we denote the optimal visitation cost of n robots, starting from some point $P \in \Delta$. When the triangle Δ is clear from the context, we abbreviate $R_n(\Delta, P)$ simply by $R_n(P)$.

In this work we are interested in determining visitation cost trade-offs with respect to different fleet sizes. In particular, for some triangle $\Delta \in \mathscr{D}$ (which is a compact set as a subest of $\mathbb{R}^2$), and for $1 \le n < m \le 3$, we define

$$\mathscr{R}_{n,m}(\Delta) := \max_{P \in \Delta} \frac{R_n(\Delta, P)}{R_m(\Delta, P)}.$$

Our main technical results pertain to the study of $\mathscr{R}_{n,m}(\Delta)$ as Δ ranges over all non-obtuse triangles $\mathscr{D}$. In particular, we determine $\inf_{\Delta \in \mathscr{D}} \mathscr{R}_{n,m}(\Delta)$ and $\sup_{\Delta \in \mathscr{D}} \mathscr{R}_{n,m}(\Delta)$ for all pairs $(n,m) \in \{(1,3),(2,3),(1,2)\}$. Our contributions are summarized in Table 1.[1]

For establishing the claims above, we observe that $\inf_{\Delta \in \mathscr{D}} \max_{P \in \Delta} \frac{R_n(\Delta,P)}{R_m(\Delta,P)} = \alpha$ is equivalent to that

$$\forall \Delta \in \mathscr{D}, \exists P \in \Delta, \frac{R_n(\Delta, P)}{R_m(\Delta, P)} \ge \alpha \quad \text{and} \quad \forall \epsilon > 0, \exists \Delta \in \mathscr{D}, \forall P \in \Delta, \frac{R_n(\Delta, P)}{R_m(\Delta, P)} \le \alpha + \epsilon.$$

[1] Note that the entries in column 1 are not obtained by multiplying the entries of columns 2,3. This is because the triangles that realize the inf and sup values are not the same in each column.

	$\mathscr{R}_{1,3}(\Delta)$	$\mathscr{R}_{2,3}(\Delta)$	$\mathscr{R}_{1,2}(\Delta)$
$\inf_{\Delta\in\mathscr{D}}$	$\sqrt{10}$	$\sqrt{2}$	2.5
$\sup_{\Delta\in\mathscr{D}}$	4	2	3

Table 1: Our main contributions.

Similarly, $\sup_{\Delta\in\mathscr{D}} \max_{P\in\Delta} \frac{R_n(\Delta,P)}{R_m(\Delta P)} = \beta$ is equivalent to that

$$\forall \Delta \in \mathscr{D}, \forall P \in \Delta, \frac{R_n(\Delta,P)}{R_m(\Delta,P)} \le \beta \quad \text{and} \quad \forall \epsilon > 0, \exists \Delta \in \mathscr{D}, \exists P \in \Delta, \frac{R_n(\Delta,P)}{R_m(\Delta,P)} \ge \beta - \epsilon.$$

Therefore, as a byproduct of our analysis, we also determine the best and the worst triangle cases of ratios $\mathscr{R}_{n,m}(\Delta)$, as well as the starting points that determine these ratios. In particular we show that (i) the extreme values of $\mathscr{R}_{1,3}(\Delta)$ are attained as Δ ranges between "thin" isosceles and equilateral triangles, and the worst starting point is the incenter, (ii) the extreme values of $\mathscr{R}_{2,3}(\Delta)$ are attained as Δ ranges between right isosceles and equilateral triangles, and the worst starting point is again the incenter, and (iii) the extreme values of $\mathscr{R}_{1,2}(\Delta)$ are attained as Δ ranges between equilateral and right isosceles triangles, and the worst starting point is the middle of the shortest altitude.

2.2 Basic Terminology & Some Useful Observations

The length of segment AB is denoted by $\|AB\|$. An arbitrary non-obtuse triangle will be usually denoted by $\triangle ABC$, which we assume is of bounded size. More specifically, without loss of generality, we often consider $\triangle ABC$ represented in the Cartesian plane in *standard analytic form*, with $A = (p,q)$, $B = (0,0)$ and $C = (1,0)$.

The cost of optimally visiting a collection of line segments $\mathscr{C}$ (triangle edges) with 1 robot starting from point P is denoted by $d(P,\mathscr{C})$. For example, when $\mathscr{C} = \{AB, BC\}$ we write $d(P,\{AB,BC\})$. When, for example, $\mathscr{C} = \{AB\}$ is a singleton set, we slightly abuse the notation and for simplicity write $d(P,AB)$ instead of $d(P,\{AB\})$. Note that if the projection P' of P onto the line defined by points A, B lies in segment AB, then $d(P,AB) = \|PP'\|$, and otherwise $d(P,AB) = \min\{\|PA\|, \|PB\|\}$. The following observation follows immediately from the definitions, and the fact that we restrict our study to non-obtuse triangles.

Observation 1 *For any non-obtuse triangle* $\Delta = \triangle ABC$, *and* $P \in \Delta$, *we have*
(i) $R_3(\Delta,P) = \max\{d(P,AB), d(P,BC), d(P,CA)\}$.
(ii) $R_2(\Delta,P) = \min \left\{ \begin{array}{l} \max\{d(P,AB), d(P,\{BC,CA\})\} \\ \max\{d(P,BC), d(P,\{AB,CA\})\} \\ \max\{d(P,CA), d(P,\{BC,AB\})\} \end{array} \right\}$
(iii) $R_1(\Delta,P) = d(P,\{AB,BC,CA\})$.

Motivated by our last observation, we also introduce notation for the cost of *ordered visitations*. Starting from point P, we may need to visit an *ordered list* of (2 or 3) line segments in a specific order. For example, we write $d(P,[AB,BC,AC])$ for the optimal cost of visiting the list of segments $[AB,BC,AC]$, in this order, with 1 robot. As we will be mainly concerned with $\triangle ABC$ edge visitations, and due to the already introduced standard analytic form, we refer to the trajectory realizing $d(P,[AB,BC,AC])$ as the (optimal) *LDR strategy* (L for "Left" edge AB, D for "Down" edge BC, and R for "Right" edge AC). We introduce analogous terminology for the remaining 5 permutations of the edges, i.e. LRD, RLD, RDL, DRL, DLR. Note that it may happen that in an optimal ordered visitation, robot visits a vertex of the triangle edges. In such a case we interpret the visitation order of the incident edges arbitrarily. For ordered visitation of 2 edges, we introduce similar terminology pertaining to (optimal) LD, LR, RL, RD, DR and DL strategies.

In order to obtain the results reported in Table 1, it is necessary to subdivide any triangle Δ into sets of points that admit the same optimal ordered visitations (e.g. all points P in which an optimal $R_1(\Delta, P)$ strategy is LRD). For $n \in \{2,3\}$ robots, the subdivision is also with respect to the cost $R_n(\Delta, P)$. Specifically for $n = 2$, the subdivision is also with respect to whether the cost $R_2(\Delta, P)$ is determined by the robot that is visiting one or two edges (see Observation 1). We will refer to these subdivisions as the R_1, R_2, R_3 *regions*. For each $n \in \{1,2,3\}$, the R_n regions will be determined by collection (loci) of points between neighbouring regions that admit more than one optimal ordered visitations.

Angles are read counter-clockwise, so that for example for $\triangle ABC$ in standard analytic form, we have $\angle A = \angle BAC$. For aesthetic reasons, we may abuse notation and drop symbol $\angle$ from angles when we write trigonometric functions. Visitation trajectories will be denoted by a list of points $\langle A_1, \ldots, A_n \rangle$ ($n \geq 2$), indicating a movement along line segments between consecutive points. Hence, the cost of such trajectory would be $\sum_{i=2}^{n} \|A_i A_{i-1}\|$.

3 Preliminary Results

3.1 Optimal Visitations of Two Triangle Edges

As a preparatory step, first consider the simpler problem of visiting two distinguished edges of a triangle $\Delta = ABC$, starting from a point within the triangle.

When $\angle A \geq \pi/3$, we define the concept of its *optimal bouncing subcone*, which is defined as a cone of angle $3\angle A - \pi$ and tip A, so that $\angle A$ and the subcone have the same angle bisector. When $\angle A = \pi/3$, then the optimal bouncing subcone is a ray with tip A that coincides with the angle bisector of $\angle A$. Whenever $\angle A < \pi/3$ we define its optimal bouncing subcone as the degenerate empty cone. The following two observations are used repeatedly in our results.

Observation 2 *If P is in the optimal bouncing subcone of $\angle A$, then $d(P, \{AB, AC\}) = \|PA\|$.*

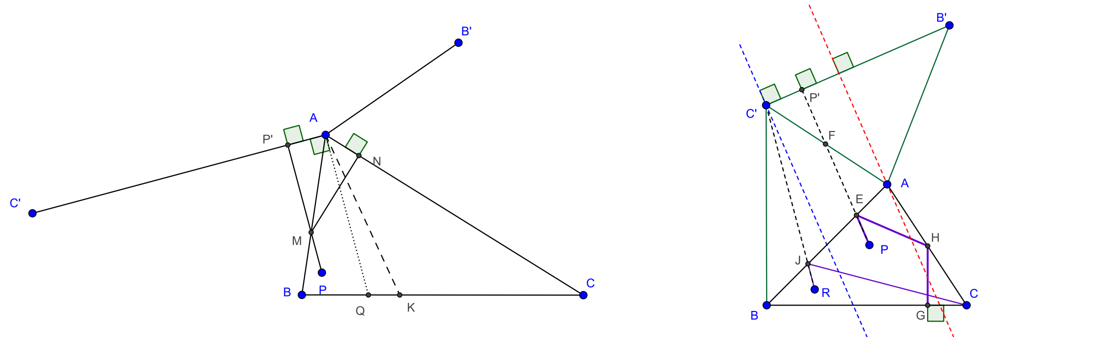
For a point $P \in \triangle ABC$ outside the optimal bouncing subcone of $\angle A$, we define the (two) *optimal bouncing points M, N of the ordered $[AB, AC]$ visitation* as follows. Let C' be the reflections of C around AB. Let also P' be the projection of P onto AC'. Then, M is the intersection of PP' with AB and N is the projection of M onto AC. Note that equivalently, M, N are determined uniquely by requiring that (i) $\angle BMP = \angle NMA$, and (ii) $\angle ANM = \pi/2$ (see also Figure 1a).

Observation 3 *If P is outside the optimal bouncing subcone of $\angle A$, then $d(P, \{AB, AC\}) = \|PM\| + \|MN\|$, where M, N are the optimal bouncing points of ordered $[AB, AC]$ visitation.*

3.2 Optimal (Ordered) Visitation of Three Triangle Edges

In this section we discuss optimal LRD visitations of non-obtuse $\triangle ABC$, together with optimality conditions when serving with one robot. Optimality conditions for the remaining 5 ordered visitations are obtained similarly. In order to determine the optimal LRD visititation, we obtain reflection C' of C across AB, and reflection B' of B across $C'A$, see also Figure 1b.

From C' and A, we draw a lines ϵ, ζ, both perpendicular to $C'B'$ which may (or may not) intersect $\triangle ABC$. We refer to line ϵ as the *LRD bounce indicator line*. We also refer to line ζ as the *LRD subopt indicator line*. Each of the lines identify a halfspace on the plane. The halfspace associated with ϵ on the side of vertex A will be called the *positive halfspace of the LRD bounce indicator line* or in short the *positive LRD bounce halfspace*, and its complement will be called the *negative LRD bounce halfspace*. The halfspace associated with ζ on the side of vertex B will be called the *positive halfspace of the LRD*



(a) Optimal trajectory for visiting $\{AB, AC\}$ for $\angle A \geq \pi/3$, starting outside the optimal bouncing subcone.

(b) Arbitrary non-obtuse $\triangle ABC$ shown with its LRD bounce indicator line and its LRD subopt indicator line.

Fig. 1

subopt indicator line, or in short the *positive LRD subopt halfspace*, and its complement will be called the *negative LRD subopt halfspace*.

For a point P in the positive LRD bounce and subopt halfspaces, let P' be its projection onto $C'B'$. Let E, F be the intersections of PP' with AB, AC', respectively. Let also H be the reflection of F across AB. Points E, H, G will be called the *optimal LRD bouncing points* for point P. The points are also uniquely determined by requiring that $\angle BEP = \angle HEA$ and that HG is perpendicular to BC. For a point R in the negative LRD bounce halfspace and in the positive subopt halfspace, let J be the intersection of RC' with AB. Point J will be called the *degenerate optimal LRD bouncing point*, which is also uniquely determined by the similar bouncing rule $\angle BJR = \angle CJA$. Finally, let A', A'' be the projection of A onto $B'C', BC$, respectively.

The next lemma refers to such points P, R together with the construction of Figure 1b. Its proof uses the observations of Section 3.1 and follows easily by noticing that the optimal LRD visitation is in 1-1 correspondence with the optimal visitation of segment $B'C'$ using a trajectory that passes from segment AB.

Lemma 1. *The optimal LRD visitation trajectory, with starting points P, R, T, is:*

- *trajectory $\langle P, E, H, G \rangle$, provided that P is in the positive LRD bounce and subopt halfspaces,*
- *trajectory $\langle R, J, C \rangle$, provided that R is in the negative LRD bounce halfspace and in the positive subopt halfspace,*
- *trajectory $\langle T, A, A'' \rangle$, provided that T is in the negative LRD subopt halfspace.*

4 Computing the R_n Regions, $n = 1, 2, 3$

By Observations 2, 3 and Lemma 1, we see that optimal visitations of 2 or 3 edges have cost equal to (i) the distance of the starting point to a line (reflection of some triangle edge), or (ii) the distance of the starting point to some point (triangle vertex) or (iii) the distance of the starting point to some triangle vertex plus the length of some triangle altitude. In this section we describe the R_n regions of certain triangles, $n \in \{1, 2, 3\}$. For this, we compare optimal ordered strategies, and the subdivisions of the regions are determined by loci of points that induce ordered trajectories of the same cost. As these costs are of type (i), (ii), or (iii) above (and considering all their combinations) the loci of points in which two ordered strategies have the same cost will be either some line (line bisector or angle bisector), or some conic section (parabola or hyperbola).

4.1 Triangle Visitation with 3 Robots - The R_3 Regions

Consider $\Delta \in \mathscr{D}$ with vertices A, B, C. For every $P \in \Delta$, any trajectories require time at least the maximum distance of P from all edges, in order to visit all of them. This bound is achieved by having all robots moving along the projection of P onto the 3 edges, and so we have $R_3(P) = \max\{d(P, AB), d(P, BC), d(P, CA)\}$, as also in Observation 1. Next we show how to subdivide the region of Δ with respect to which of the 3 projections is responsible for the optimal visitation cost. For this, we let I denote the incenter (the intersection of angle bisectors) of Δ. Let also K, L, M be the intersections of the bisectors with edges BC, CA and AB, respectively, see also Figure 2.

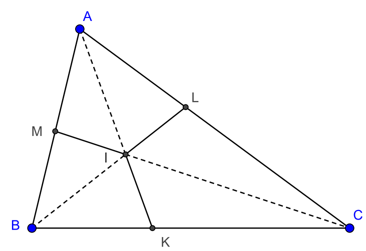


Fig. 2: The R_3 regions of an arbitrary non-obtuse $\triangle ABC$. AK, BL, CM are the angle bisectors of $\angle A, \angle B, \angle C$, respectively. Recall that the incenter I is equidistant from all triangle edges.

Lemma 2. *For every starting point $P \in \Delta$, we have that*

$$R_3(\Delta, P) = \begin{cases} d(P, AB) \text{ , provided that } P \in CLIK \\ d(P, BC) \text{ , provided that } P \in AMIL \\ d(P, CA) \text{ , provided that } P \in BKIM \end{cases}.$$

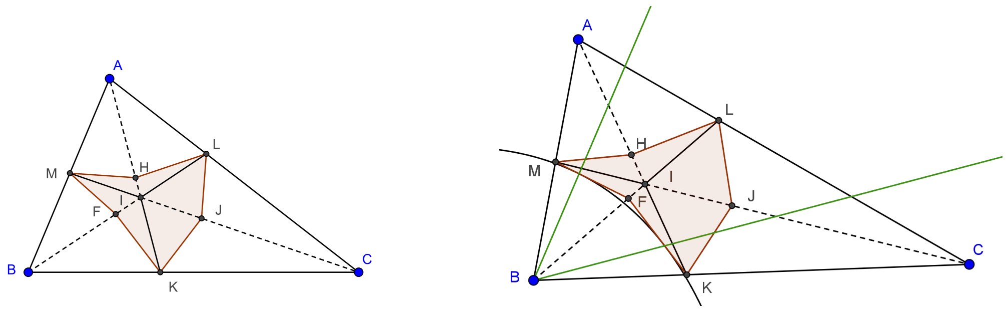
4.2 Triangle Visitation with 2 Robots - The R_2 Regions

In this section we show how to subdivide the region of any non-obtuse triangle $\Delta \in \mathscr{D}$ into subregions with respect to the optimal trajectories and their costs, for a fleet consisting of 2 robots. The following technical lemma describes a geometric construction.

Lemma 3. *Consider non-obtuse $\triangle ABC$ along with its incenter I. Let K, M be the intersections of angle bisectors of A, C with segments BC, AB respectively. From K, M we consider cones of angles A, C respectively, having direction toward the interior of the triangle, and placed so that their bisectors are perpendicular to BC, AB, respectively. Then, the extreme rays of the cones intersect at some point F in line segment BI.*

Motivated by Lemma 3, we will be referring to the subject point F in the line segment BI as the *separator of the angle B bisector*. Similarly, we obtain separators J, H of angles C, A bisectors, respectively, see also Figure 3a. In what follows, we will be referring to the (possibly non-convex) hexagon $MFKJLH$ as the R_2 *(hexagon) separator of* $\triangle ABC$.

The remaining of the section refers to non-obtuse triangle $\Delta = \triangle ABC$ as in Figure 3a, where in particular $MFKJLH$ is the R_2 separator of Δ. Assume that $\angle B \geq \pi/3$. It can be shown that for every point P either in MF or FK which are outside the optimal bouncing subcone of angle B, we have that $d(P, AC) = d(P, \{BC, AB\})$. For points within the subcone, the optimal trajectory to visit $\{BC, AB\}$ would be to go directly to B. So for points P within the optimal bouncing subcone, condition $d(P, AC) =$



(a) The R_2 hexagon separator of $\triangle ABC$.

(b) The refined R_2 mixed-hexagon separator of $\triangle ABC$, where $\angle B > \pi/3$.

Fig. 3

$d(P, \{BC, AB\})$ translates into that P is equidistant from AC and B. Hence, P lies in a parabola with AC being the directrix and B being the focus. Next, we refer to that parabola as the *separating parabola of* B.

Motivated by the previous observation, we introduce the notion of the *refined* R_2 *mixed-hexagon separator of triangle* Δ as follows. For every angle of Δ which is more than $\pi/3$, we replace the portion of the R_2 hexagon separator within the optimal bouncing subcone of the same angle by the corresponding separating parabola. In Figure 3b we display an example where only one angle is more than $\pi/3$. Combined with Observation 1 (ii), we can formalize our findings as follows.

Lemma 4. *For every starting point on the boundary of the refined* R_2 *mixed-hexagon separator of a triangle* Δ*, the cost of visiting only the opposite edge equals the cost of visiting the other two edges. For every starting point* P *outside the* R_2 *separator,* $R_2(P)$ *equals the distance of* P *to the opposite edge. Moreover, for every starting point* P *in the interior of the refined* R_2 *separator,* $R_2(P)$ *is determined by the cost of visiting two of the edges of* Δ*.*

(a) The R_2 regions of the equilateral triangle, see also Corollary 1 and Lemma 5 for detailed description.

(b) The R_2 regions of the right isosceles, see also Corollary 2 and Lemma 6 for detailed description. The coloured region identifies the refined R_2 mixed-hexagon separator.

Fig. 4

Lemma 4 implies the following corollaries pertaining to specific triangles $\triangle ABC$. In both statements, and the associated figures, I is the incenter of the triangles, and points K, L, M are defined as in Figure 2.

Corollary 1 (Hexagon separator of equilateral triangle). *Consider equilateral $\Delta = \triangle ABC$, see Figure 4a. Let W, Z, Y be the intersections of AK, BL, CM respectively (also the separators of angle A bisector, angle B bisector, and angle C bisector, respectively). Then, the R_2 hexagon separator of Δ is $MZKYLW$, which is also triangle MKL. More specifically, for all $P \in \triangle AML$, we have that $R_2(\Delta, P) = d(P, BC)$.*

Corollary 2 (Mixed-hexagon separator of right isosceles). *Consider right isosceles $\Delta = \triangle ABC$, see Figure 4b. The separator of angle A bisector is incenter I. Let also F, J be the separators of angle B bisector and angle C bisector, respectively. Then, the R_2 hexagon separator of Δ is $IMFKJL$. The parabola with directix BC and focus A, intersecting AK at Q and passing through M, L is the separating parabola of A. Hence, for every point $P \in \Delta$ above the parabola, we have $R_2(\Delta, P) = d(P, BC)$, as well as for every point X in tetragon $MBKF$, we have $R_2(\Delta, X) = d(X, AC)$.*

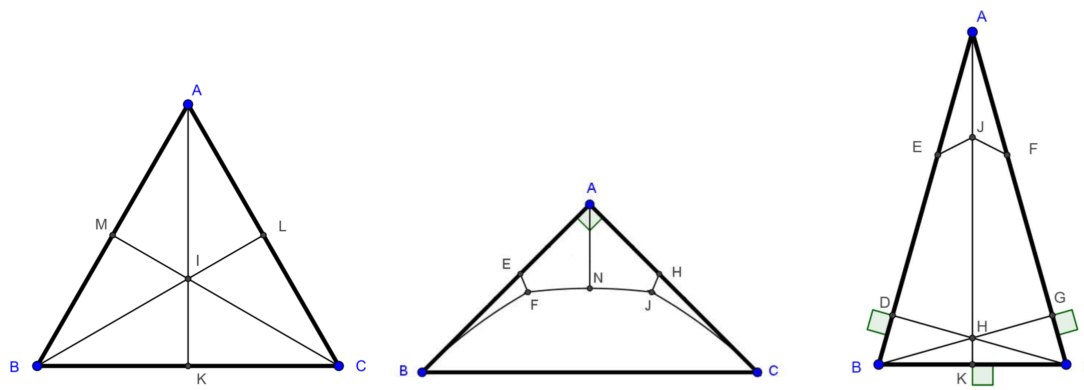
Describing the subdivisions within the refined R_2 mixed-hexagon separator for arbitrary triangles is a challenging task. On the other hand, by Observation 1 (ii) and Lemma 4 the cost within the separator is determined by the cost of visiting just two edges. Also, by Observations 2, 3 the cost of such visitation can be described either as a distance to a line or to a point. We conclude that, within the R_2 separator, the subdivisions are determined by separators that are either parts of lines or parabolas (loci of points for which the cost of visiting some two edges are equal). Hence, for any fixed triangle, an extensive case analysis pertaining to pairwise comparisons of visitations costs can determine all R_2 subdivisions (and the challenging ones are within the refined separator). In what follows we summarize formally the subdivisions only of two triangle types, focusing on the visitation cost of all starting points within the (refined) hexagon separators.

Lemma 5 (R_2 regions of an equilateral triangle). *Consider equilateral $\Delta = \triangle ABC$, as in Corollary 1, see Figure 4a. Then for every starting point $P \in \triangle MWI$, we have that $R_2(\Delta, P) = d(P, [AB, AC])$. The remaining cases of starting points within the hexagon separator $MZKYLW$ follow by symmetry.*

Lemma 6 (R_2 regions of a right isosceles triangle). *Consider right isosceles $\Delta = \triangle ABC$, as in Corollary 2, see Figure 4b. Consider parabola with directrix the line passing through B that is perpendicular to BC (also the reflection of BC across AB) and focus A, passing through M, K and intersecting BL at point T (define also S as the symmetric point of T across AK). That parabola is the locus of points P for which $\|PA\| = d(P, [AB, BC])$. Let also A' be the reflection of A across BC. Consider parabola with directrix BA' and focus A, passing through T and intersecting AK at point U. That parabola is the locus of points P for which $\|PA\| = d(P, [BC, AB])$. Therefore, if P is a starting visitation point, we have that:*
- $R_2(\Delta, P) = \|PA\|$, for all P in mixed closed shape $MTUSLQ$ (grey shape in Figure 4b),
- $R_2(\Delta, P) = d(P, [AB, BC])$, for all P in mixed closed shape MFT (blue shape in Figure 4b),
- $R_2(\Delta, P) = d(P, [BC, AB])$, for all P in mixed closed shape $FKUT$ (red shape in Figure 4b).
The visitation costs with starting points in the remaining subdivisions of the refined R_2 mixed-hexagon separator, green and purple regions in Figure 4b, follow by symmetry.

4.3 Triangle Visitation with 1 Robot - The R_1 Regions

In this section we show how to partition the region of an arbitrary non-obtuse $\triangle ABC$ into sets of points P with respect to the optimal strategy of $R_1(P)$. There are 6 possible visitation strategies for $d(P, \{AB, AC, AB\})$, one for each permutation of the edges indicating the order they are visited (ordered visitations). Clearly, it is enough to describe, for each two ordered visitations, the borderline (separator) of points in which the two visitations have the same cost. By Lemma 1, any such ordered visitation cost is the distance of the starting point either to a point, or to a line, or a distance to a line plus the length of some altitude. Since the R_1 regions are determined by separators, i.e. loci of points in which different ordered visitations induce the same costs, it follows that these separators are either lines, or conic sections. Therefore, by exhaustively pairwise-comparing all ordered visitations along with their separators, we can determine the R_1 regions of any triangle. Next, we explicitly describe the R_1 regions only for three types of triangles that we will need for our main results. For the sake of avoiding redundancies, we omit any descriptions that are implied by symmetries.



(a) The R_1 regions of an equilateral triangle.

(b) The R_1 regions of a right isosceles triangle.

(c) The R_1 regions of an isosceles triangle ABC with small $\angle A$.

Fig. 5

The next lemma describes the R_1 regions of an equilateral triangle, as in Figure 5a.

Lemma 7 **(R_1 regions of an equilateral triangle).** *Consider equilateral triangle $\triangle ABC$ with angle bisectors AK, BL, CM and incenter I. Then, the angle bisectors are the loci of points in which optimal ordered visitations have the same cost. Moreover, for every starting point $P \in \triangle AMI$, the optimal strategy of $R_1(\Delta, P)$ is LRD visitation.*

The next lemma describes the R_1 regions of a right isosceles, as in Figure 5b. Curve FJ is part of the parabola with directrix the relfection of BC across A and focus the reflection of A across BC. Curve BF is part of the parabola with directrix a line parallel to AB which is $\|AB\|$ away from AB, and focus the reflection of A across BC. Curve CJ is part of the parabola with directrix a line parallel to AC which is $\|AC\|$ away from AC, and focus the reflection of A across BC. CE (not shown in the figure) is the bisector of $\angle C$, and segment EF is part of the reflection of that bisector across AB. BH is the bisector of $\angle C$, and segment HJ is part of the reflection of that bisector across AC. Segment AN is part of the altitude corresponding to A.

Lemma 8 **(R_1 regions of a right isosceles triangle).** *Consider right isosceles $\Delta = \triangle ABC$, and starting point P. Then, the optimal visitation strategy for $R_1(\Delta, P)$ is:*
- an LRD visitation, if $P \in AEFN$,
- an LDR visitation if $P \in BFE$, and
- both an DRL, DLR visitation if $P \in BCJF$ (trajectory visits $\{AB, AC\}$ at point A).

Next we consider a "thin" isosceles $\Delta = \triangle ABC$ with $\angle A \le \pi/3$, as in Figure 5c. (Eventually we will invoke the next lemma for $\angle A \to 0$.) AK is the altitude corresponding to A. CD, BG are the altitudes corresponding to AB, AC, respectively. CE, BF (not shown) are the extreme rays of the optimal bouncing subcone corresponding to C, B, respectively. H is the intersection of AK with BG (and CD), i.e. the orthocenter of the triangle. Segment EJ (as part of a line) is the reflection of EC (as part of a line) across AB. Segment FJ (as part of a line) is the reflection of BF (as part of a line) across AC.

Lemma 9 **(R_1 regions of a thin isosceles triangle).** *Consider isosceles $\Delta = \triangle ABC$, with $\angle A \le \pi/3$ and starting point P. Then, the optimal visitation strategy for $R_1(\Delta, P)$ is:*
- an LRD visitation if $P \in AEJ$,
- both LRD and LDR (optimal strategy is to visit first AB and then move to C), if $P \in EDHJ$,
- an LDR visitation, if $P \in DBH$, and
- a DLR visitation if $P \in BKH$.

5 Visitation Trade-offs

In this section we outline how we obtain our main results, as reported in Table 1. For this we invoke the lemmata we already established, along with the the following claims (requiring lengthy and technical proofs) pertaining to optimal visitation costs of some special starting points. For the remaining of the section, we denote by I the incenter of $\triangle ABC$. All three following lemmata refer to non-obtuse $\triangle ABC$.

Lemma 10. *If $\angle C$ is the largest angle, then $R_2(I) = \|IC\|$.*

Lemma 11. *If $\angle A$ is the largest angle, then $R_1(I) = \|IA'\|$, where A' is the reflection of A across BC.*

Lemma 12. *Let $\angle A \ge \angle B \ge \angle C$, and T be the middle point of the altitude corresponding to the largest edge BC. Then the optimal $R_1(T)$ strategy is of LRD type, and has cost $\frac{1}{2}(2-\cos(2A))\sin(B)\sin(C)\csc(B+C)$.*

5.1 Searching with 1 vs 3 Robots

First we sketch the proof of $\sup_{\Delta\in\mathcal{D}} \mathcal{R}_{1,3}(\Delta) = 4$. The lower bound for $\sup_{\Delta\in\mathcal{D}} \mathcal{R}_{1,3}(\Delta)$ is given by the following lemma that utilizes Lemma 2 and Lemma 11.

Lemma 13. *Let Δ be an equilateral triangle.Then, $R_1(I)/R_3(I) = 4$.*

The remaining of the section is devoted in proving a tight upper bound for $\sup_{\Delta\in\mathcal{D}} \mathcal{R}_{1,3}(\Delta)$. Without loss of generality, we also assume that the starting point P lies within the tetragon (4-gon) $AMIL$, see also Figure 2.

In order to provide the promised upper bound, we propose a heuristic upper bound for $R_1(P)$, as follows. Consider the projections P_1, P_2, P_3 of P onto AB, BC and CA respectively. Then, three (possibly) suboptimal visitation trajectories for one robot are $T_C(P) := \langle P, P_1, P, C, \rangle$, $T_A(P) := \langle P, P_2, P, A \rangle$, $T_B(P) := \langle P, P_3, P, B \rangle$, that is $R_1(P) \le \min\{T_A(P), T_B(P), T_C(P)\}$. The upper bound proof follows by following lemma.

Lemma 14. *If $\angle A \le \pi/3$, then $\min\{T_B(P), T_C(P)\}/R_3(P) \le 4$. If $\angle A \ge \pi/3$, then $T_A(P)/R_3(P) \le 4$.*

Next we outline how we obtain that $\inf_{\Delta\in\mathcal{D}} \mathcal{R}_{1,3}(\Delta) = \sqrt{10}$. First, using Lemma 2 and Lemma 9 we show that $\inf_{\Delta\in\mathcal{D}} \mathcal{R}_{1,3}(\Delta) \le \sqrt{10}$.

Lemma 15. *For isosceles ABC with base BC, we have $\lim_{\angle A\to 0} \max_{P\in ABC} \frac{R_1(P)}{R_3(P)} = \sqrt{10}$.*

Next, we invoke Lemma 2 and Lemma 11 in order to show that $\inf_{\Delta\in\mathcal{D}} \mathcal{R}_{1,3}(\Delta) \ge \sqrt{10}$.

Lemma 16. *For any triangle $\Delta \in \mathcal{D}$ we have $R_1(I)/R_3(I) \ge \sqrt{10}$.*

5.2 Searching with 2 vs 3 Robots

First we outline the proof of that $\sup_{\Delta\in\mathscr{D}} \mathscr{R}_{2,3}(\Delta) = 2$. For this, and using Lemma 2 and Lemma 10, we establish that $\sup_{\Delta\in\mathscr{D}} \mathscr{R}_{2,3}(\Delta) \geq 2$.

Lemma 17. *For the equilateral triangle we have* $R_2(I)/R_3(I) = 2$.

The remaining of the section is devoted in proving that $\sup_{\Delta\in\mathscr{D}} \mathscr{R}_{2,3}(\Delta) \leq 2$. In that direction, we consider a triangle $\Delta = ABC$ along with its incenter I, see also Figure 2.

In order to provide the promised upper bound, we propose a heuristic upper bound for $R_2(P)$. The two robots visit all edges as follows; one robot goes to the vertex corresponding to the largest angle (visiting the two incident edges), and the second robot visits the remaining edge moving along the projection of P along that edge. The heuristic is used the to show the following.

Lemma 18. *For any* $\Delta \in \mathscr{D}$ *and starting point* P*, we have* $R_2(P)/R_3(P) \leq 2$.

Next we outline how we prove that $\inf_{\Delta\in\mathscr{D}} \mathscr{R}_{2,3}(\Delta) = \sqrt{2}$. Using Lemma 2 and Lemma 6 we can show that $\inf_{\Delta\in\mathscr{D}} \mathscr{R}_{2,3}(\Delta) \leq \sqrt{2}$.

Lemma 19. *Let* $\triangle ABC$ *be a right isosceles. Then, we have* $\max_{P\in ABC} \frac{R_2(P)}{R_3(P)} = \sqrt{2}$.

Then, by invoking Lemma 10 we show that $\inf_{\Delta\in\mathscr{D}} \mathscr{R}_{2,3}(\Delta) \geq \sqrt{2}$.

Lemma 20. *For any* $\Delta \in \mathscr{D}$*, we have* $R_2(I)/R_3(I) \geq \sqrt{2}$.

5.3 Searching with 1 vs 2 Robots

Finally, we outline how we prove that $\sup_{\Delta\in\mathscr{D}} \mathscr{R}_{1,2}(\Delta) = 3$. Using a simple heuristic upper bound for R_1, we can show the following.

Lemma 21. *For any* $\Delta \in \mathscr{D}$ *and any starting point* $P \in \Delta$*, we have* $R_1(P)/R_2(P) \leq 3$.

The lower bound for $\sup_{\Delta\in\mathscr{D}} \mathscr{R}_{1,2}(\Delta)$ is attained for the right isosceles triangle (and for certain starting point). Indeed, using Lemma 12 we show that $\sup_{\Delta\in\mathscr{D}} \mathscr{R}_{1,2}(\Delta) \geq 3$.

Lemma 22. *Let* ABC *be a right isosceles triangle with right angle* A*. Let also* P *be the middle point of the altitude corresponding to angle* A*. Then,* $R_1(P)/R_2(P) = 3$.

It remains to sketch the proof of $\inf_{\Delta\in\mathscr{D}} \mathscr{R}_{1,2}(\Delta) = 5/2$. For this, using Lemma 5 and Lemma 7 we prove that $\inf_{\Delta\in\mathscr{D}} \mathscr{R}_{1,2}(\Delta) \leq 5/2$.

Lemma 23. *For the equilateral triangle* Δ*, we have* $\max_{P\in\Delta} R_1(P)/R_2(P) = 5/2$.

Then, using Lemma 12, we prove that $\inf_{\Delta\in\mathscr{D}} \mathscr{R}_{1,2}(\Delta) \geq 5/2$.

Lemma 24. *For any* $\triangle ABC \in \mathscr{D}$*, let* T *be the middle point of the altitude corresponding to the largest edge. Then, we have* $R_1(T)/R_2(T) \geq 5/2$.

6 Conclusions

We considered a new vehicle routing-type problem in which (fleets of) robots visit all edges of a triangle. We proved tight bounds regarding visitation trade-offs with respect to the size of the available fleet. In order to avoid degenerate cases of visiting the edges with 3 robots, we only focused our study on non-obtuse triangles. The case of arbitrary triangles, as well as of other topologies, e.g. graphs, remains open. We believe the definition of our problem is of independent interest, and that the study of efficiency trade-offs in combinatorial problems with respect to the number of available processors (that may not be constant as in our case), e.g. vehicle routing type problems, will lead to new, deep and interesting questions.

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