THE SURVIVING RATE OF PLANAR GRAPHS

PAWEŁ PRAŁAT

The following *firefighter problem* on a finite graph $G = (V, E)$ was introduced by Hartnell at the conference in 1995 [3]. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit, a firefighter protects one vertex which is not yet on fire, and then fire spreads to all unprotected neighbours of the vertices on fire. (Once a vertex is on fire or gets protected it stays in such state forever.) Since the graph is finite, at some point each vertex is either on fire or is protected by the firefighter, and the process is finished. (Alternatively, one can stop the process when no neighbour of the vertices on fire is unprotected. The fire will no longer spread.) The objective of the firefighter is to save as many vertices as possible. Today, 15 years later, our knowledge about this problem is much greater and a number of papers have been published. We would like to refer the reader to the survey of Finbow and MacGillivray for more information [6].

We would like to focus on the following property. Let $\mathrm{sn}(G, v)$ denote the number of vertices in G the firefighter can save when a fire breaks out at vertex $v \in V$, assuming the best strategy is used. The surviving rate $\rho(G)$ of G, introduced in [5], is defined as the expected percentage of vertices that can be saved when a fire breaks out at a random vertex of G (uniform distribution is used), that is, $\rho(G) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v)$. It is not difficult to see that for cliques $\rho(K_n) = \frac{1}{n}$, since no matter where a fire breaks out only one vertex can be saved. For paths we get that

$$\rho(P_n) = \frac{1}{n^2} \sum_{v \in V} \mathrm{sn}(G, v) = \frac{1}{n^2} \left(2(n-1) + (n-2)(n-2) \right) = 1 - \frac{2}{n} + \frac{2}{n^2}$$

(one can save all but one vertex when a fire breaks out at one of the leaves; otherwise two vertices are burned). It is not surprising that a path can be easily protected, and in fact, all trees have this property. Cai, Cheng, Verbin, and Zhou [1] proved that the greedy strategy of Hartnell and Li [4] for trees saves at least $1 - \Theta(\log n / n)$ percentage of vertices on average for an n-vertex tree. Moreover, they managed to prove that for every outerplanar graph G, $\rho(G) \geq 1 - \Theta(\log n / n)$. Both results are asymptotically tight and improved earlier results of Cai and Wang [2]. Let us note that there is no hope for similar result for planar graphs, since, for example, $\rho(K_{2,n}) = 2/(n+2) = o(1)$.

Let us stay focused on sparse graphs. It is clear that sparse graphs are easier to control so their survival rates should be relatively large. Finbow, Wang, and Wang [7] showed that any graph G with average degree strictly smaller than 8/3 has the surviving rate bounded away from zero. Formally, it has been shown that any graph G with $n \geq 2$ vertices and $m \leq (\frac{4}{3} - \varepsilon)n$ edges satisfies $\rho(G) \geq \frac{6\varepsilon}{5} > 0$, where $0 < \varepsilon < \frac{5}{24}$ is a fixed number. This result was recently improved by the author of this extended abstract to show that any graph G with average degree strictly smaller than 30/11 has the surviving rate bounded away from zero [8].

Theorem 1 ([8]). *Suppose that graph G has $n \geq 2$ vertices and $m \leq (\frac{15}{11} - \varepsilon)n$ edges, for some $0 < \varepsilon < \frac{1}{2}$. Then, $\rho(G) \geq \frac{\varepsilon}{60}$.*

(Let us note that the goal was to show that the surviving rate is bounded away from zero, not to show the best lower bound for $\rho(G)$. The constant $\frac{1}{60}$ can be easily improved with more careful calculations.)

On the other hand there are some dense graphs with large survival rates (take, for example, a large collection of cliques). However, in [8] a construction of a sparse random graph on n vertices with the survival rate tending to zero as n goes to infinity is proposed. Hence the result is tight and the constant $\frac{15}{11}$ cannot be improved.

It would be nice to find the threshold for other families of graphs, including planar graphs.

Question 1. *Determine the largest real number M such that every planar graph G with $n \geq 2$ vertices and $m \leq (M - \varepsilon)n$ edges has $\rho(G) \geq c \cdot \varepsilon$ for some $c > 0$.*

It follows from Theorem 1 and the fact that $\rho(K_{2,n}) = o(1)$ that $\frac{15}{11} \leq M \leq 2$.

The second question was asked in [7].

Question 2. *Determine the least integer g^* such that there is a constant $0 < c < 1$ such that every planar graph G with girth at least g^* has $\rho(G) \geq c$.*

Note that a connected planar graph with n vertices and girth g can have at most $\frac{g}{g-2}(n-2)$ edges (see, for example, [7]). Thus, from Theorem 1 it follows that $g^* \leq 8$. Using the fact that $\rho(K_{2,n}) = o(1)$ one more time, we conclude that $5 \leq g^* \leq 8$.

References

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DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, WV 26506-6310, USA

E-mail address: `pralat@math.wvu.edu`