Mathematical Logic for Computer Science

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Answers to Exercises

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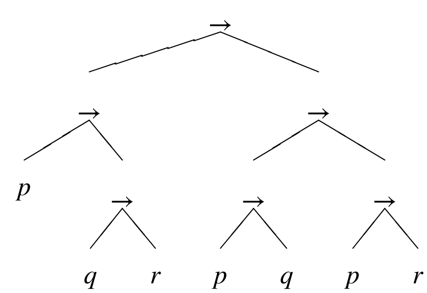
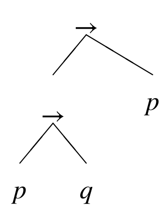
1 Introduction

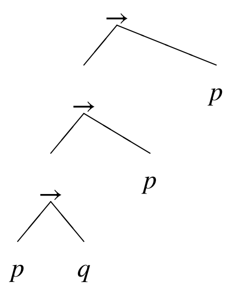
1. 'Some' is being used both as an existential quantifer and as an instantiation. Symbolically, the first statement is $\exists x(car(x) \wedge rattles(x))$, and the second is $car(mycar)$. Clearly, $rattles(mycar)$ does not follow.

2 Propositional Calculus: Formulas, Models, Tableaux

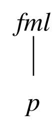
1. The truth tables can be obtained from the Prolog program. Here are the formation trees.



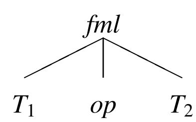




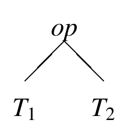
2. The proof is by an inductive construction that creates a formation tree from a derivation tree. Let *fml* be a nonterminal node with no occurrences of of *fml* below it. If the node is



for some atom p, then the only formation tree is p itself. Otherwise, suppose that the non-terminal is



where T_1 and T_2 are formation trees and *op* is a binary operator. The only fomration tree is



3. The proof is by induction on the structure of an arbitrary formula A. If A is an atom, there is no difference between an assignment and an interpretation. If $A = A_1 \, op \, A_2$ is a formula with a binary operator, then by the inductive hypothesis $v(A_1)$ and $v(A_1)$ are uniquely defined, so there is a single value that can be assigned to $v(A)$ according to the table. The case for negation is similar.

   Induction is also used to prove that assignments that agree on the atoms of a formula A agree on the formula. For an atom of A, the claim is trivial, and the inductive step is straightforward.

4. Construct the truth tables for the formulas and compare that they are the same. For example, the table for the formulas in the fourth equivalence is:

A	B	$v(A \rightarrow B)$	$v(A \wedge \neg B)$	$v(\neg(A \wedge \neg B))$
T	T	T	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	T

where we have added an extra column for the subformula $A \wedge \neg B$.

5. By associativity and idempotence, $((p \oplus q) \oplus q) \equiv (p \oplus (q \oplus q)) \equiv p \oplus \mathit{false}$. Using the definition of $\oplus$, we find that $p \oplus \mathit{false} \equiv p$. Similarly, $((p \leftrightarrow q) \leftrightarrow q) \equiv (p \leftrightarrow (q \leftrightarrow q)) \equiv p \leftrightarrow \mathit{true} \equiv p$.

6. We prove

$$A_1 \mathbin{op} A_2 \equiv B_1 \circ \cdots \circ B_n \equiv \neg \cdots \neg B_i$$

by induction on n. If $n = 1$, clearly B_i is either A_1 or A_2. If $n = 2$ and the definition of $\circ$ is

A_1	A_2	$A_1 \circ A_2$
T	T	F
T	F	T
F	T	F
F	F	T

then $A_1 \circ A_2 \equiv \neg A_2,\ A_2 \circ A_1 \equiv \neg A_1,\ A_1 \circ A_1 \equiv \neg A_1,\ A_2 \circ A_2 \equiv \neg A_2$, and symmetrically for

A_1	A_2	$A_1 \circ A_2$
T	T	F
T	F	F
F	T	T
F	F	T

Suppose now that

$$A_1 \mathbin{op} A_2 \equiv (B_1 \circ \cdots \circ B_k) \circ (B_{k+1} \circ \cdots B_n).$$

By the inductive hypothesis, $B_1 \circ \cdots \circ B_k \equiv \neg \cdots \neg B_i$ and $B_{k+1} \circ \cdots \circ B_n \equiv \neg \cdots \neg B_{i'}$, where $\neg \cdots \neg B_i$ and $\neg \cdots \neg B_{i'}$ are each logically equivalent to A_1, $\neg A_1$, A_2, or $\neg A_2$. By an argument similar to that used for $n = 2$, the claim follows.

7. Let A be a formula constructed only from only p and $\wedge$ or $\vee$. We prove by induction that $A \equiv p$. Clearly, if A is an atom, A is p. Suppose that A is $A_1 \wedge A_2$ and that $A_1 \equiv A_2 \equiv p$. Then $A \equiv p \wedge p \equiv p$. Similarly, for $\vee$.

8. If $U = \{p\}$ and B is $\neg p$, then U is satisfiable, but $U \cup \{B\} = \{p, \neg p\}$ is not.

9. We prove Theorem 2.35; the others are similar. Suppose that $U - \{A_i\}$ is satisfiable and let $\mathcal{I}$ be a model. But a valid formula is true in all interpretations, so A is true in $\mathcal{I}$. Thus, $\mathcal{I}$ is a model for U, contradicting the assumption.

10.

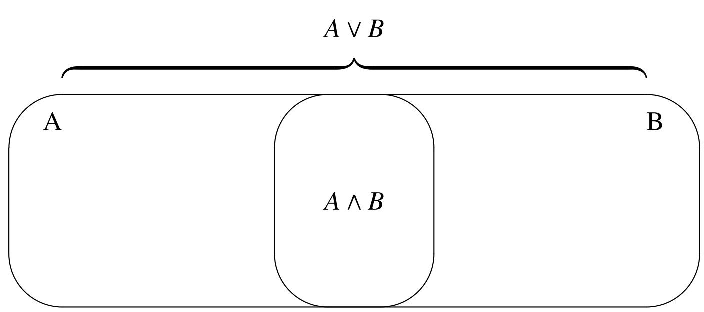
    Theorem 2.38: Any interpretation which falsifies $U = \{A_1, \ldots, A_n\}$ assigns true to $A_1 \wedge \cdots \wedge A_n \rightarrow A$ by the definition of $\wedge$ and $\rightarrow$. Any model for $U = \{A_1, \ldots, A_n\}$, assigns true to A by assumption.

    Theorem 2.39: Adding an additional assumption can only *reduce* the number of interpretations that have to satisfy A.

    Theorem 2.40: Since any interpretation satisfies a valid formulas, the set of models for U is exactly the set of models for $U - \{B\}$.

11. For $\mathcal{T}(U)$ to be closed under logical consequence means that if $\{A_1, \ldots, A_n\} \models A$ where $A_i \in \mathcal{T}(U)$ then $A \in \mathcal{T}(U)$. Let $\mathcal{I}$ be an arbitrary model for U. If $A_i \in \mathcal{T}(U)$, then $U \models A_i$, so $\mathcal{I}$ is a model for A_i for all i and $\mathcal{I}$ is a model for A. Thus $U \models A$ so $A \in \mathcal{T}(U)$.

12. A logical equivalence can be proven by replacing $\equiv$ by $\leftrightarrow$ and constructing a truth table or semantic tableau for the negation. Alternatively, truth tables can be constructed for both sides and checked for equality. The use of Venn diagrams is interesting in that it shows an equivalence between propositional logical and set theory or Boolean algebra. The Venn diagram for a proposition represents the set of interpretations for which it is true as demonstrated in the following diagram:

    How is $A \rightarrow B$ represented in a Venn diagram? $A \rightarrow B$ is logically equivalent to $\neg A \vee B$, so the diagram for $A \rightarrow B$ consists of the area *outside* A together with the area for B. $A \leftrightarrow B$ is represented by the intersection of the areas for A and B (where both are true), together with the area outside both (where both are false).

    Here are the proofs of the equivalences in terms of Venn diagrams:

    $A \rightarrow B \equiv A \leftrightarrow (A \wedge B)$: If a point is in the area for $A \rightarrow B$, it is either in the area for $A \wedge B$ or outside both. So if it is in A, it must be within $A \wedge B$.

$A \rightarrow B \equiv B \leftrightarrow (A \vee B)$: Similarly, if it is in the union of the areas, it must be within B.

$A \wedge B \equiv (A \leftrightarrow B) \leftrightarrow (A \vee B)$: Points are in both of the areas for $A \leftrightarrow B$ and $A \vee B$ iff they are within the area for $A \wedge B$.

$A \leftrightarrow B \equiv (A \vee B) \rightarrow (A \wedge B)$: If a point is in the union of the areas for A and B, it must be within the area for $A \wedge B$ if it is to be within the area for $A \leftrightarrow B$.

13. Let $W(l) = 4^{e(l)+1} \cdot (3b(l) + n(l) + 3)$, where $e(l)$ is the number of equivalence and non-equivalence operators. If $e(l)$ is decreased by one, the new $b(l)$ will be $2b(l) + 2$ and the new $n(l)$ will be at most $2n(l) + 2$. A computation will show that:
$$4^{e(l)+1} \cdot (3b(l) + n(l) + 3) > 4^{e(l)} \cdot (6b(l) + 2n(l) + 8).$$

14. We have to show that if the label of a node contains a complementary pair of formulas, then any tableau starting from that node will close (atomically). The proof is by induction. The base case is trivial. Suppose that $\{\alpha, \neg\, \alpha\} \subseteq U(n)$, and that we use the α-rule on α, resulting in $\{\alpha_1, \alpha_2, \neg\, \alpha\} \subseteq U(n')$, and then the β-rule on $\neg\, \alpha$, resulting in $\{\alpha_1, \alpha_2, \neg\, \alpha_1\} \subseteq U(n''_1)$ and $\{\alpha_1, \alpha_2, \neg\, \alpha_2\} \subseteq U(n''_2)$. The result follows by the inductive hypothesis. The case for $\{\beta, \neg\, \beta\} \subseteq U(n)$ is similar.

15. Add facts to the `alpha` and `beta` databases for the decompositions on page 32.

16. A node can become closed only if the addition of a new subformula to the label contradicts an existing one. Rather than check all elements of the label against all others, include the check for contradiction in the predicates `alpha_rule` and `beta_rule`.

3 Propositional Calculus: Deductive Systems

1.

1.	$A, B, \neg\, A$	Axiom
2.	$\neg\, B, B, \neg\, A$	Axiom
3.	$\neg(A \rightarrow B), B, \neg\, A$	$\beta\!\rightarrow$ 1, 2
4.	$\neg(A \rightarrow B), \neg\, \neg\, B, \neg\, A$	$\alpha\, \neg$ 3
5.	$\neg(A \rightarrow B), (\neg\, B \rightarrow \neg\, A)$	$\alpha\!\rightarrow$ 4
6.	$(A \rightarrow B) \rightarrow (\neg\, B \rightarrow \neg\, A)$	$\alpha\!\rightarrow$ 5

1.	$A, \neg A, B$	Axiom
2.	$A, \neg B, B$	Axiom
3.	$A, \neg(\neg A \rightarrow B), B$	$\beta\rightarrow$ 1, 2
4.	$\neg B, \neg A, B$	Axiom
5.	$\neg B, \neg B, B$	Axiom
6.	$\neg B, \neg(\neg A \rightarrow B), B$	$\beta\rightarrow$ 4, 5
7.	$\neg(A \rightarrow B), \neg(\neg A \rightarrow B), B$	$\beta\rightarrow$ 3, 6
8.	$\neg(A \rightarrow B), (\neg A \rightarrow B) \rightarrow B$	$\alpha\rightarrow$ 7
9.	$(A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)$	$\alpha\rightarrow$ 8

1.	$\neg A, B, A$	Axiom
2.	$A \rightarrow B, A$	$\alpha\rightarrow$ 1
3.	$\neg A, A$	Axiom
4.	$\neg((A \rightarrow B) \rightarrow A), A$	$\beta\rightarrow$ 2, 3
5.	$((A \rightarrow B) \rightarrow A) \rightarrow A$	$\alpha\rightarrow$ 4

2. The proof is by induction on the structure of the proof. If $\vdash U$ where U is an axiom, then U is a set of literals containing a complementary pair $\{p, \neg p\}$, that is, $U = U_0 \cup \{p, \neg p\}$. Obviously, there is a closed tableau for $\bar{U} = \bar{U}_0 \cup \{\neg p, p\}$.

   Let the last step of the proof of U be an application an α- or β-rule to obtain a formula $A \in U$; we can write $U = U_0 \cup \{A\}$. In the following, we use $\vee$ and $\wedge$ as examples for α- and β-formulas.

   Case 1: An α-rule was used on $U' = U_0 \cup \{A_1, A_2\}$ to prove $U = U_0 \cup \{A_1 \vee A_2\}$. By the inductive hypothesis, there is a closed tableau for $\bar{U}' = \bar{U}_0 \cup \{\neg A_1, \neg A_2\}$. Using the tableau α-rule, there is a closed tableau for $\bar{U} = \bar{U}_0 \cup \{\neg(A_1 \vee A_2)\}$.

   Case 2: An β-rule was used on $U' = U_0 \cup \{A_1\}$ and $U'' = U_0 \cup \{A_2\}$ to prove $U = U_0 \cup \{A_1 \wedge A_2\}$. By the inductive hypothesis, there are closed tableaux for $\bar{U}' = \bar{U}_0 \cup \{\neg A_1\}$ and $\bar{U}'' = \bar{U}_0 \cup \{\neg A_2\}$. Using the tableau β-rule, there is a closed tableau for $\bar{U} = \bar{U}_0 \cup \{\neg(A_1 \wedge A_2)\}$.

3.

1.	$\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$	Theorem 3.24
2.	$\vdash (A \rightarrow B)$	Assumption
3.	$\vdash \neg B \rightarrow \neg A$	MP 1, 2
4.	$\vdash \neg B$	Assumption
5.	$\vdash A$	MP 3, 4

4.

1.	$\neg A, \neg B, A$	Axiom
2.	$\neg A, B \rightarrow A$	$\alpha\rightarrow$ 1
3.	$A \rightarrow (B \rightarrow A)$	$\alpha\rightarrow$ 2

   For Axiom 2 we will use a shortcut by taking as an axiom any set of *formulas* containing a complementary pair of literals.

1.	$B, A, \neg A, C$	Axiom
2.	$B, \neg B, \neg A, C$	Axiom
3.	$B, \neg(A \rightarrow B), \neg A, C$	$\beta\rightarrow$ 1, 2
4.	$\neg C, \neg(A \rightarrow B), \neg A, C$	Axiom
5.	$\neg(B \rightarrow C), \neg(A \rightarrow B), \neg A, C$	$\beta\rightarrow$ 3, 4
6.	$A, \neg(A \rightarrow B), \neg A, C$	Axiom
7.	$\neg(A \rightarrow (B \rightarrow C)), \neg(A \rightarrow B), \neg A, C$	$\beta\rightarrow$ 5, 6
8.	$\neg(A \rightarrow (B \rightarrow C)), \neg(A \rightarrow B), A \rightarrow C$	$\alpha\rightarrow$ 7
9.	$\neg(A \rightarrow (B \rightarrow C)), (A \rightarrow B) \rightarrow (A \rightarrow C)$	$\alpha\rightarrow$ 8
10.	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$	$\alpha\rightarrow$ 9

The proof of Axiom 3 is similar to the proof of $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ from exercise 1.

5.

1.	$\{\neg A \rightarrow A\} \vdash \neg A \rightarrow A$	Assumption
2.	$\{\neg A \rightarrow A\} \vdash \neg A \rightarrow \neg\neg A$	Contrapositive
3.	$\{\neg A \rightarrow A\} \vdash (\neg A \rightarrow \neg\neg A) \rightarrow \neg\neg A$	Theorem 3.28
4.	$\{\neg A \rightarrow A\} \vdash \neg\neg A$	MP 2, 3
5.	$\{\neg A \rightarrow A\} \vdash A$	Double negation
6.	$\vdash (\neg A \rightarrow A) \rightarrow A$	Deduction

6.

1.	$\vdash A \rightarrow (\neg A \rightarrow B)$	Theorem 3.21
2.	$\vdash A \rightarrow (A \vee B)$	Definition of $\vee$

1.	$\vdash B \rightarrow (B \vee A)$	Just proved
2.	$\vdash B \rightarrow (A \vee B)$	Theorem 3.32

The proof of Theorem 3.32 does not use this theorem so it can be used here.

1.	$\vdash (\neg C \rightarrow (A \rightarrow B)) \rightarrow ((\neg C \rightarrow A) \rightarrow (\neg C \rightarrow B))$	Axiom 2
2.	$\vdash (C \vee (A \rightarrow B)) \rightarrow ((C \vee A) \rightarrow (C \vee B))$	Definition of $\vee$
3.	$\vdash (A \rightarrow B) \rightarrow (C \vee (A \rightarrow B))$	Just proved
4.	$\vdash (A \rightarrow B) \rightarrow ((C \vee A) \rightarrow (C \vee B))$	Transitivity

7. Of course, $\leftrightarrow$ should be $\rightarrow$.

1.	$\{(A \vee B) \vee C\} \vdash (A \vee B) \vee C$	Assumption
2.	$\{(A \vee B) \vee C\} \vdash \neg(\neg A \rightarrow B) \rightarrow C$	Definition of $\vee$
3.	$\{(A \vee B) \vee C\} \vdash \neg C \rightarrow (\neg A \rightarrow B)$	Contrapositive
4.	$\{(A \vee B) \vee C\} \vdash \neg A \rightarrow (\neg C \rightarrow B)$	Exchange
5.	$\{(A \vee B) \vee C\} \vdash (\neg C \rightarrow B) \rightarrow (\neg B \rightarrow C)$	Contrapos., double neg.
6.	$\{(A \vee B) \vee C\} \vdash \neg A \rightarrow (\neg B \rightarrow C)$	Transitivity
7.	$\{(A \vee B) \vee C\} \vdash A \vee (B \vee C)$	Definition of $\vee$
8.	$\vdash (A \vee B) \vee C \rightarrow A \vee (B \vee C)$	Deduction

8. The proofs are trivial.

9. The second node below is obtained by applying the α-rule for $\rightarrow$ three times.

$$
\begin{tikzcd}[column sep=small]
 & & \neg((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))) \arrow[d] & & & \\
 & & A \rightarrow (B \rightarrow C), A \rightarrow B, A, \neg C \arrow[dl] \arrow[dr] & & & \\
 & A \rightarrow (B \rightarrow C), \neg A, A, \neg C & & A \rightarrow (B \rightarrow C), B, A, \neg C \arrow[dl] \arrow[dr] & & \\
 & \times & \neg A, B, A, \neg C & & B \rightarrow C, B, A, \neg C \arrow[dl] \arrow[dr] & \\
 & & \times & \neg B, B, A, \neg C & & C, B, A, \neg C \\
 & & & \times & & \times
\end{tikzcd}
$$

10. Let $(A_1, \ldots, A_n)$ be the elements of $U - U'$ in some order.

1.	$\vdash \bigvee U'$	Assumption
2.	$\vdash \bigvee U' \vee A_1$	Theorem 3.31
	$\cdots$	
n+1.	$\vdash \bigvee U' \vee A_1 \vee \cdots \vee A_n$	Theorem 3.31

So we have to prove that if U' is a permutation of U and $\vdash \bigvee U$ then $\vdash \bigvee U'$. The proof is by induction on n the number of elements in U. If $n = 1$, there is nothing to prove, and if $n = 2$, the result follows immediately from Theorem 3.32. Let $\bigvee U = \bigvee U_1 \vee \bigvee U_2$ and $\bigvee U' = \bigvee U_1' \vee \bigvee U_2'$ have n elements. If U_1' and U_2' are permutations of U_1 and U_2, respectively, then the result follows by the inductive hypothesis and Theorem 3.31. Otherwise, without loss of generality, suppose that there is an element A of U_2' which is in U_1. Suppose that $\bigvee U_2' = A \vee \bigvee U_2''$, so that $\bigvee U' = \bigvee U_1' \vee (A \vee \bigvee U_2'')$. Then by Theorem 3.33, $\bigvee U' = (\bigvee U_1' \vee A) \vee \bigvee U_2''$. Thus all we have to prove is that $A_1 \vee \cdots \vee A_i \vee \cdots \vee A_k$ can be written $A_i \vee A_1 \vee \cdots \vee A_{i-1} \vee A_{i+1} \vee \cdots \vee A_k$ for arbitrary i. This is proved by a simply induction using Theorem 3.33.

11. The first formula was proved in Theorem 3.24.

1.	$\{A \rightarrow B, \neg A \rightarrow B\} \vdash \neg A \rightarrow B$	Assumption
2.	$\{A \rightarrow B, \neg A \rightarrow B\} \vdash \neg B \rightarrow A$	Contrapositive
3.	$\{A \rightarrow B, \neg A \rightarrow B\} \vdash A \rightarrow B$	Assumption
4.	$\{A \rightarrow B, \neg A \rightarrow B\} \vdash \neg B \rightarrow B$	Transitivity
5.	$\{A \rightarrow B, \neg A \rightarrow B\} \vdash (\neg B \rightarrow B) \rightarrow B$	Theorem 3.29
6.	$\{A \rightarrow B, \neg A \rightarrow B\} \vdash B$	MP 4, 5
7.	$\{A \rightarrow B\} \vdash (\neg A \rightarrow B) \rightarrow B$	Deduction
8.	$\vdash (A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)$	Deduction

1.	$\{(A \to B) \to A\} \vdash (A \to B) \to A$	Assumption
2.	$\{(A \to B) \to A\} \vdash \neg A \to (A \to B)$	Theorem 3.20
3.	$\{(A \to B) \to A\} \vdash \neg A \to A$	Transitivity
4.	$\{(A \to B) \to A\} \vdash (\neg A \to A) \to A$	Theorem 3.29
5.	$\{(A \to B) \to A\} \vdash A$	MP 3, 4
6.	$\vdash (A \to B) \to A) \to A$	Deduction

12. The deduction theorem can be used because its proof only uses Axioms 1 and 2.

1.	$\{\neg B \to \neg A, A\} \vdash \neg B \to \neg A$	Assumption
2.	$\{\neg B \to \neg A, A\} \vdash (\neg B \to \neg A) \to ((\neg B \to A) \to B)$	Axiom 3'
3.	$\{\neg B \to \neg A, A\} \vdash (\neg B \to A) \to B$	MP 1,2
4.	$\{\neg B \to \neg A, A\} \vdash A \to (\neg B \to A)$	Axiom 1
5.	$\{\neg B \to \neg A, A\} \vdash A$	Assumption
6.	$\{\neg B \to \neg A, A\} \vdash \neg B \to A$	MP 4,5
7.	$\{\neg B \to \neg A, A\} \vdash B$	MP 6,3
8.	$\{\neg B \to \neg A\} \vdash A \to B$	Deduction
9.	$\vdash (\neg B \to \neg A) \to (A \to B)$	Deduction

13. It follows from Definition 3.47 that a sequent $\{U_1, \ldots, U_n\} \Rightarrow \{V_1, \ldots, V_m\}$ is true iff $\neg U_1 \vee \cdots \vee \neg U_n \vee V_1 \vee \cdots \vee V_m$ is true. The completeness of $\mathcal{S}$ follows from the completeness of $\mathcal{G}$ by showing that the rules of the two are the same. For example,

$$\frac{U \cup \{A\} \Rightarrow V \cup \{B\}}{U \Rightarrow V \cup \{A \to B\}}$$

is

$$\frac{\neg U_1 \vee \cdots \neg U_m \vee \neg A \vee V_1 \vee \cdots \vee V_n \vee B}{\neg U_1 \vee \cdots \neg U_m \vee V_1 \vee \cdots \vee V_n \vee (A \to B)}$$

which is the α-rule for $\to$, and

$$\frac{U \Rightarrow V \cup \{A\} \qquad U \cup \{B\} \Rightarrow V}{U \cup \{A \to B\} \Rightarrow V}$$

is

$$\frac{\neg U_1 \vee \cdots \neg U_m \vee V_1 \vee \cdots \vee V_n \vee A \qquad \neg U_1 \vee \cdots \neg U_m \vee \neg B \vee V_1 \vee \cdots \vee V_n}{\neg U_1 \vee \cdots \neg U_m \vee \neg(A \to B) \vee V_1 \vee \cdots \vee V_n}$$

which is the β-rule for $\to$. We leave the check of the other rules to the reader.

14. If $\vdash \neg A_1 \vee \cdots \vee \neg A_n$ then clearly $U \vdash \neg A_1 \vee \cdots \vee \neg A_n$, since we do not need to use the assumptions. But that is the same as $U \vdash A_1 \to (A_2 \to \cdots \to \neg A_n)$. Now $U \vdash A_i$ is trivial, so by $n-1$ applications of modus ponens, $U \vdash \neg A_n$, which together with $U \vdash A_n$, prove that U is inconsistent.

Conversely, if U is inconsistent, then $U \vdash A$ and $U \vdash \neg A$ for some A. But only a finite number of formulas $\{A_1, \ldots, A_n\} \subseteq U$ are used in either one of the proofs, so $\{A_1, \ldots, A_n\} \vdash A$ and $\{A_1, \ldots, A_n\} \vdash \neg A$. By n applications of the deduction theorem, $\vdash A_1 \to \cdots \to A_n \to \neg A$ and $\vdash A_1 \to \cdots \to A_n \to A$. From propositional reasoning, $\vdash A_1 \to \cdots \to A_n \to$ *false*, and $\vdash \neg A_1 \vee \cdots \vee \neg A_n$.

15.

(a) Assume that $U \subseteq S$ is finite and unsatisfiable. Then $\neg \bigwedge U$ is valid, so $\vdash \neg \bigwedge U$ by completeness (Theorem 3.35). By repeated application of $\vdash A \to (B \to (A \wedge B))$ (Theorem 3.30), $U \vdash \bigwedge U$. But $S \cup U = S$ and you can always add unused assumptions to a proof so $S \vdash \neg \bigwedge U$ and $S \vdash \bigwedge U$, contradicting the assumption that S is consistent.

(b) Assume that $S \cup \{A\}$ and $S \cup \{\neg A\}$ are both inconsistent. By Theorem 3.41, both $S \vdash A$ and $S \vdash \neg A$, so S is inconsistent, contradicting the assumption.

(c) S can be extended to a maximally consistent set.

Consider an enumeration $\{A_1, \ldots, \}$ of all propositional formulas. Let $S_0 = S$ and for each i let $S_{i+1} = S_i \cup \{A_i\}$ or $S_{i+1} = S_i \cup \{\neg A_i\}$, whichever is consistent. (We have just proved that one of them must be.) Let $S' = \bigcup_{i=0}^{\infty} S_i$. First we prove that S' is consistent. If S' is inconsistent, then $S' \vdash p \wedge \neg p$ (for example) by Theorem 3.39. But only a finite number of elements of S' are used in the proof, so there must be some large enough i such that S_i includes them all. Then $S_i \vdash p \wedge \neg p$ contracting the consistency of S_i. To prove that S' is maximally consistent, suppose that $B \notin S'$. By construction, $\neg B \in S_i \subset S'$ for some i, so $S' \cup \{B\} \vdash \neg B$. But trivially, $S' \cup \{B\} \vdash B$, so $S' \cup \{B\}$ is inconsistent.

16. I would be pleased if someone would contribute a program!

4 Propositional Calculus: Resolution and BDDs

1. This theorem can be proved both syntactically and semantically. The syntactic proof uses the same construction as the one for CNF except that the distributive laws used are: $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ and $(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$. For the semantic proof, start by constructing a truth table for the formula A. For each line in the truth table that evaluates to T, construct a conjunction with the literal p if p is assigned T in that row and $\bar{p}$ if p is assigned F. Let A' be the disjunction of all these conjunctions. Then $A \equiv A'$. Let v be an arbitrary model for A, that is, $v(A) = T$. Then the assignments in v are such that the row of the truth table contains T, and by construction $v(C) = T$ for the conjunction C built from that row. Since A' is a disjunction of such conjunctions, it is sufficient for one of them to be true for A' to be true. Hence $v(A') = T$. For the converse, if v is an arbitrary interpretation so that $v(A') = T$, then by the structure of A', $v(C) = T$ for at least one conjunction C in A' (in fact, for exactly one such conjunction). But for this assignment, $v(A) = T$ by construction.
2. The formula constructed in the previous exercise is in complete DNF.

3. I would be pleased if someone would contribute a program!

4. This exercise is rather trivial because the sets of clauses are satisfiable and for S a set of satisfiable clauses, $S \approx \{\}$ the valid empty set of clauses. For each of the sets, we give a sequence of sets obtained by using the various lemmas.

$$
\begin{aligned}
&\{p\bar{q},\ q\bar{r},\ rs,\ p\bar{s}\} \approx \{q\bar{r},\ rs\} \approx \{q\bar{r}\} \approx \{\,\},\\
&\{pqr,\ \bar{q},\ p\bar{r}s,\ qs,\ p\bar{s}\} \approx \{pr,\ p\bar{r}s,\ s,\ p\bar{s}\} \approx \{pr,\ p\} \approx \{p\} \approx\approx \{\,\},\\
&\{pqrs,\ \bar{q}rs,\ \bar{p}rs,\ qs,\ \bar{p}s\} \approx \{\bar{q}rs,\ \bar{p}rs,\ qs,\ \bar{p}s\} \approx \{\,\},\\
&\{\bar{p}q,\ qrs,\ \bar{p}\bar{q}rs,\ \bar{r},\ q\} \approx \{\bar{p}rs,\ \bar{r}\} \approx \{\bar{r}\} \approx \{\,\}.
\end{aligned}
$$

5.

Refutation 1:

5.	$\bar{q}r$	1, 2
6.	r	3, 5
7.	$\square$	4, 6

Refutation 2:

5.	$\bar{p}\bar{q}$	1, 4
6.	p	2, 4
7.	$\bar{q}$	5, 6
8.	q	3, 4
9.	$\square$	7, 8

6. The clausal form of the set is:

$$(1)\,p,\ (2)\,\bar{p}qr,\ (3)\,\bar{p}\bar{q}\bar{r},\ (4)\,\bar{p}st,\ (5)\,\bar{p}\bar{s}\bar{t},\ (6)\,\bar{s}q,\ (7)\,rt,\ (8)\,\bar{t}s.$$

A refutation is:

9.	$\bar{p}\bar{s}r$	5, 7
10.	$\bar{p}s$	4, 8
11.	$\bar{p}\bar{q}\bar{s}$	3, 9
12.	$\bar{p}\bar{s}$	11, 6
13.	$\bar{p}$	12, 10
14.	$\square$	13, 1

7. The clausal form of the formulas is:

$$\{(1)\,\bar{s}\bar{b}1\bar{b}2,\ (2)\,\bar{s}b1b2,\ (3)\,s\bar{b}1b2,\ (4)\,sb1\bar{b}2,\ (5)\,\bar{c}b1,\ (6)\,\bar{c}b2,\ (7)\,c\bar{b}1\bar{b}2\}.$$

The addition of the set of clauses: $\{(8)\,b1,\ (9)\,b2,\ (10)\,\bar{s},\ (11)\,\bar{c}\}$, enables a refutation to be done by resolving clauses $11, 7, 8, 9$. The addition of the clauses $\{(8)\,b1,\ (9)\,b2,\ (10)\,\bar{s},\ (11)\,c\}$ gives a satisfiable set by assigning F to s and T to all other atoms (check!). The meaning of the satisfiable set is that $1 \oplus 1 = 0$ carry 1, by identifying 1 with T and 0 with F. The unsatisfiable set shows that it is not true that $1 \oplus 1 = 0$ carry 0.

8. The statement of the claim should say: Prove adding a unit clause to a set of clauses *such that the atom of the unit clause does not already appear in the set* and ....

   Let S be the original set of clauses and S' the new set of clauses obtained by adding $\{l\}$ to S and l^c to every other clause in S, and let v be a model for S. Extend v to v' by defining $v'(p) = T$ if $l = p$, $v'(p) = F$ if $l^c = p$, and v' is the same as v on all other atoms. (Here is where we need the proviso on the new clause.) By construction, $v(l) = T$ so the additional clause in S' is satisfied. For every other clause C, $v(C) = v'(C) = T$ since the addition of a literal to a clause (which is a disjunction) cannot falsify it.

9. By induction on the depth of the resolution tree. If the depth of the tree is 1, the result is immediate from Theorem 4.24. If the depth of the tree is n, then the children of the root are satisfiable by the inductive hypothesis, so the root is satisfiable by Theorem 4.24.

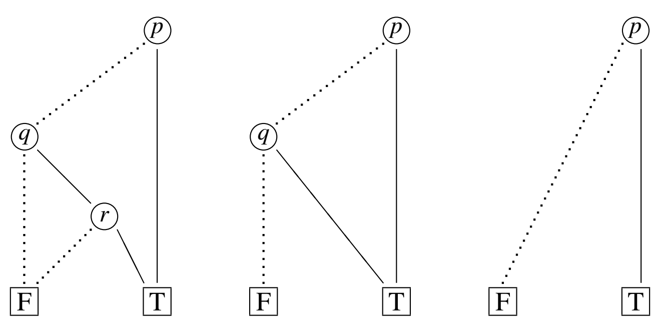
10. First prove a lemma: for any v, $v(A_1|_{p=T}\ op\ A_2|_{p=T}) = v(A_1\ op\ A_2)$ if $v(p) = T$ and $v(A_1|_{p=F}\ op\ A_2|_{p=F}) = v(A_1\ op\ A_2)$ if $v(p) = F$. The proof is by structural induction. Clearly, $v(p|_{p=T}) = T = v(p)$ and $q|_{p=T}$ is the same formula as q for $q \neq p$. Suppose now that $v(p) = T$. By the inductive hypothesis, $v(A_1|_{p=T}) = v(A_1)$ and $v(A_2|_{p=T}) = v(A_2)$ so by the semantic definitions of the operators, $v(A_1|_{p=T}\ op\ A_2|_{p=T}) = v(A_1\ op\ A_2)$. A similar argument holds for F.

    We can now prove the Shannon expansion. Let v be an arbitrary interpretation. If $v(p) = T$,

    $$v(\,(p \wedge (A_1|_{p=T}\ op\ A_2|_{p=T})) \vee (\neg p \wedge (A_1|_{p=F}\ op\ A_2|_{p=F}))\,) = v(A_1|_{p=T}\ op\ A_2|_{p=T}),$$

    which equals $v(A_1\ op\ A_2)$ by the lemma, and similarly if $v(p) = F$. Since v was arbitrary, the formulas are logically equivalent.

11. From Example 4.62, the BDDs for $p \vee (q \wedge r)$ and for $A|_{r=T}$ and $A|_{r=F}$ are:



    Using the algorithm `apply` with $\vee$ gives the middle BDD above for $p \vee q$: recursing on the left subBDD gives $q \vee F$ which is q and recursing on the right subBDD is clearly T. Using the algorithm `apply` with $\wedge$ gives the right BDD above for p: recursing on the left subBDD gives the controlling operand F for $\wedge$ and recursing on the right subBDD is clearly T.

12. The programs in the software archive implement the optimizations.

13. Let us number the clauses as follows:

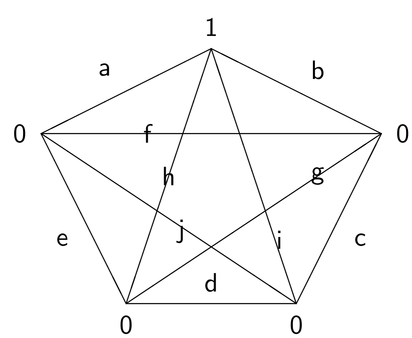
$$(1)\,\bar{p}q,\ (2)\,p\bar{q},\ (3)\,prs,\ (4)\,\bar{p}\bar{r}s,\ (5)\,\bar{p}r\bar{s},\ (6)\,p\bar{r}\bar{s},$$
$$(7)\,\bar{s}t,\ (8)\,s\bar{t},\ (9)\,\bar{q}rt,\ (10)\,q\bar{r}t,\ (11)\,qr\bar{t},\ (12)\,\bar{q}\bar{r}\bar{t}.$$

The refutation is not for the faint-hearted....

13.	qrs	1, 3
14.	qst	13, 10
15.	$\bar{q}\bar{r}s$	2, 4
16.	$\bar{q}st$	15, 9
17.	st	14, 16
18.	s	17, 8
19.	$q\bar{r}\bar{s}$	1, 6
20.	$q\bar{s}\bar{t}$	19, 11
21.	$\bar{q}r\bar{s}$	2, 5
22.	$\bar{q}\bar{s}\bar{t}$	21, 12
23.	$\bar{s}\bar{t}$	20, 22
24.	$\bar{s}$	23, 7
25.	$\Box$	18, 24

14. This is trivial as all the leaves are labeled false.

15. Here is the complete graph with the edges and vertices labeled.



The clauses are the eight even-parity clauses on the atoms *abhi* and the eight odd-parity clauses on each of *aefj*, *degh*, *cdij* and *bcfg*, forty clauses in all. We leave the construction of a resolution refutation to the reader.

16. Suppose $\Pi(v_n) = b_n$ and let $C = l_1 \cdots l_k$ be an arbitrary clause associated with n. Then C can is only falsified by the assignment

$$v(p_i) = F \text{ if } l_i = p_i \;\text{ and }\; v(p_i) = T \text{ if } l_i = \bar{p}_i.$$

Then

$$\begin{array}{lcr} \Pi(C) & = & \text{(by definition)} \\ \text{parity of negated atoms of } C & = & \text{(by construction)} \\ \text{parity of literals assigned } T & = & \text{(by definition)} \\ \Pi(v_n) & = & \text{(by assumption)} \\ b_n, & & \end{array}$$

which contradicts the assumption that $C \in C(n)$. Thus if $\Pi(v_n) = b_n$, v must satisfy all clauses in $C(n)$.

17. Decision procedure for satisfiability of *sets* of formulas whose only operators are $\neg$, $\leftrightarrow$ and $\oplus$:

- Use $p \oplus q \equiv \neg(p \leftrightarrow q) \equiv \neg p \leftrightarrow q$ and $\neg\neg p \equiv p$ to reduce the formulas to the form $q_1 \leftrightarrow \cdots \leftrightarrow q_n$.
- Use commutativity, associativity and:
  $p \leftrightarrow p \equiv \mathit{true}$, $p \leftrightarrow \neg p \equiv \mathit{false}$, $p \leftrightarrow \mathit{true} \equiv p$, $p \leftrightarrow \mathit{false} \equiv \neg p$,
  to reduce the formulas to equivalences on distinct atoms.
- If all formulas reduce to *true*, the set is valid. If some formula reduces to *false*, the set is unsatisfiable.
- Otherwise, delete all formulas which reduce to *true*. Transform the formulas as in the lemma. Let $\{p_1, \ldots, p_m\}$ be the new atoms. Assign *true* to p_1 and each q_j^1 to which it is equivalent. By induction, assign *true* to p_i, unless some q_j^i has already been assigned to; if so, assign its value to p_i. If p_i has already been assigned a clashing value, the set of formulas is unsatisfiable.

Each of the three steps increases the size of the formula by a small polynomial, so the procedure is efficient.

5 Predicate Calculus: Formulas, Models, Tableaux

1. A falsifying interpretation is $(\{1,2\}, \{\{2\}\}, \{1\})$. Then $v(p(2)) = T$ so $v(\exists\, x p(x)) = T$, but $v(p(a)) = v(p(1)) = F$.

2. We will prove the validity by constructing a closed semantic tableau for the negation of each formula. To simply formatting, a linear representation will be