RYERSON UNIVERSITY
DEPARTMENT OF MATHEMATICS

MTH714 - LOGIC & COMPUTABILITY - MIDTERM TEST

October 20, 2008

INSTRUCTIONS

1. Duration: 1.5 hours
2. You are allowed one 8.5” $\times$ 11” formula sheet (two-sided).
3. Marks (out of 30) are shown in brackets.
4. Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there.
5. Try to provide full justification for your answers.
6. Do not separate the sheets.
7. Have your student card available on your desk.

Last Name (Print): ____________________

First Name (Print): ____________________

Student I.D. ______________________

Signature ______________________

Grade /30

[6 marks] **(2)** (a) Show that the set of connectives $\{\rightarrow, false\}$ is adequate, where *false* is a constant function whose value is always F (false). [Hint: It is sufficient to show that $\neg$ can be expressed in terms of $\rightarrow$ and *false*.]

Solution: Notice, first, that

$$p \rightarrow false \equiv \neg p$$

and this suffices to show that the set $\{\rightarrow, false\}$ is adequate, since $\{\rightarrow, \neg\}$ is an adequate set of connectives.

(b) Explain why the set of connectives $\{\wedge, \rightarrow\}$ is not adequate.

Solution: If $A(p)$ is a formula which has p as its only atom and whose connectives are from the set $\{\wedge, \rightarrow\}$, then it is easy to see that, when $v(p) = T$, we must have $v(A) = T$.

For that reason, it is impossible to have

$$A \equiv \neg p$$

which shows that the set $\{\wedge, \rightarrow\}$ is not adequate.

[6 marks] **(3)** Using the method of semantic tableaux, show that the formula

$$(p \to q) \to ((\neg p \to q) \to q)$$

is valid.

Solution:

$$\begin{array}{ccccc}
 & & \neg[(p \to q) \to ((\neg p \to q) \to q)] & & \\
 & & | & & \\
 & & p \to q,\ \neg[(\neg p \to q) \to q] & & \\
 & & | & & \\
 & & p \to q,\ \neg p \to q,\ \neg q & & \\
 & / & & & \backslash \\
 & \neg p,\ \neg p \to q,\ \neg q & & & \neg q,\ \neg p \to q,\ q \\
 / & | & & & \\
 \neg p,\ p,\ \neg q & \neg p,\ q,\ \neg q & & &
\end{array}$$

Since all the leaves of the tableau are closed, the formula

$$\neg[(p \to q) \to ((\neg p \to q) \to q)]$$

is unsatisfiable. Therefore,

$$(p \to q) \to ((\neg p \to q) \to q)$$

is valid.

[6 marks] **(4)** Consider the proof of the valid formula

$$A \to (B \to (A \wedge B))$$

in the Hilbert's proof system $\mathcal{H}$

Step	Formula	Justification
1.	$\{A, B\} \vdash (A \to \neg B) \to (A \to \neg B)$	Theorem from class ($\vdash A \to A$)
2.	$\{A, B\} \vdash A \to ((A \to \neg B) \to \neg B)$	Exchange of Hypotheses Rule 1
3.	$\{A, B\} \vdash A$	Assumption
4.	$\{A, B\} \vdash (A \to \neg B) \to \neg B$	MP 3,2
5.	$\{A, B\} \vdash \neg\neg B \to \neg(A \to \neg B)$	Contrapositive Rule 4
6.	$\{A, B\} \vdash B$	Assumption
7.	$\{A, B\} \vdash \neg\neg B$	Double Negation Rule 6
8.	$\{A, B\} \vdash \neg(A \to \neg B)$	MP 7,5
9.	$\{A\} \vdash B \to \neg(A \to \neg B)$	Deduction Rule 8
10.	$\vdash A \to (B \to \neg(A \to \neg B))$	Deduction Rule 9
11.	$\vdash A \to (B \to (A \wedge B))$	Definition of $\wedge$

Provide justification for each step in this proof. You may use any rule proved or stated in lectures.

[6 marks] **(5)** Convert the following formula into a CNF:

$$((A \to \neg B) \to (C \to \neg A)) \to (\neg B \to \neg C)$$

Solution:

$$\begin{gathered}
((A \to \neg B) \to (C \to \neg A)) \to (\neg B \to \neg C) \\
\equiv ((\neg A \lor \neg B) \to (\neg C \lor \neg A)) \to (B \lor \neg C) \\
\equiv (\neg(\neg A \lor \neg B) \lor (\neg C \lor \neg A)) \to (B \lor \neg C) \\
\equiv ((A \land B) \lor (\neg C \lor \neg A)) \to (B \lor \neg C) \\
\equiv \neg((A \land B) \lor (\neg C \lor \neg A)) \lor (B \lor \neg C) \\
\equiv (\neg(A \land B) \land \neg(\neg C \lor \neg A)) \lor (B \lor \neg C) \\
\equiv ((\neg A \lor \neg B) \land (C \land A)) \lor (B \lor \neg C) \\
\equiv (\neg A \lor \neg B \lor B \lor \neg C) \land (C \lor B \lor \neg C) \land (A \lor B \lor \neg C)
\end{gathered}$$

[The simplest possible expression for this CNF is $A \lor B \lor \neg C$.]

[6 marks] **(6)** Using resolution, determine whether the following set of clauses is satisfiable or not:

$$\{\overline{p}q, pr, \overline{q}s, \overline{r}s, \overline{s}\}$$

Solution:

$$S_0 = \{\overline{p}q, pr, \overline{q}s, \overline{r}s, \overline{s}\}$$

The literals $\overline{r}s$ and $\overline{s}$ are clashing and $Res(\overline{r}s, \overline{s}) = \overline{r}$. Then,

$$S_1 = \{\overline{p}q, pr, \overline{q}s, \overline{r}s, \overline{s}, \overline{r}\}$$

The literals $\overline{r}$ and $\overline{p}r$ are clashing, and $Res(\overline{p}r, \overline{r}) = \overline{p}$ and

$$S_2 = \{\overline{p}q, pr, \overline{q}s, \overline{r}s, \overline{s}, \overline{r}, \overline{p}\}$$

Now, the literals pr and $\overline{p}$ are clashing and $Res(pr, \overline{p}) = r$ so we get

$$S_3 = \{\overline{p}q, pr, \overline{q}s, \overline{r}s, \overline{s}, \overline{r}, \overline{p}, r\}$$

Finally, the literals r and $\overline{r}$ are clashing and $Res(r, \overline{r}) = \Box$, which proves the unsatisfiability of the original set of clauses.