RYERSON UNIVERSITY
DEPARTMENT OF MATHEMATICS

MTH714 - LOGIC & COMPUTABILITY - MIDTERM TEST

October 26, 2007

INSTRUCTIONS

1. Duration: 2 hours
2. You are allowed one 8.5” $\times$ 11” formula sheet (two-sided).
3. Marks (out of 30) are shown in brackets.
4. Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there.
5. Try to provide full justification for your answers.
6. Do not separate the sheets.
7. Have your student card available on your desk.

Last Name (Print): ______________________________

First Name (Print): ______________________________

Student I.D. ______________________________

Signature ______________________________

Grade /30

[5 marks] **(1)** For each of the following two statements either explain why it is true or give a counterexample:

(a) If $A \to B$ is a valid formula (i.e. a tautology), and A is valid, then B is valid.

(b) If $A \to B$ is satisfiable and A is satisfiable, then B is satisfiable.

(c) If $A \to B$ is valid and A is satisfiable, then B is satisfiable.

Solution:
(a) The statement is true. Suppose A and $A \to B$ are both valid formulas. If v is any truth assignment,

$$v(A) = v(A \to B) = T$$

so:

$$v(B) = T$$

which means that B is also valid.

(b) This statement is false. As a counterexample, take e.g.

$$A = p, \qquad B = p \wedge \neg p$$

Clearly, A is satisfiable and $A \to B = p \to (p \wedge \neg p)$ is satisfiable (take the assignment $v(p) = F$). However, B is a contradiction.

(c) This statement is true. Suppose $A \to B$ is valid and A is satisfiable. Then, for some truth assignment v, $v(A) = T$. But, since $v(A \to B) = T$, we also have $v(B) = T$, so B is satisfiable, too.

[5 marks] **(2)** (a) Show that the single connective $\{\downarrow\}$ is adequate, where $\downarrow$ is the NOR operation, given by the truth table

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

(b) Explain why the set of connectives $\{\vee, \leftrightarrow\}$ is not adequate.

Solution:
(a) Notice that

$$p \downarrow p \equiv \neg p$$

Also,

$$p \vee q \equiv \neg(p \downarrow q) \equiv (p \downarrow q) \downarrow (p \downarrow q)$$

Since $\{\neg, \vee\}$ is an adequate set of connectives and all of its operations are expressible in $\{\downarrow\}$, the latter set of connectives is also adequate.

(b) Suppose

$$\neg p \equiv A(p, \vee, \leftrightarrow)$$

where $A(p, \vee, \leftrightarrow)$ is a formula that uses $\vee$ and $\leftrightarrow$ as its only connectives. It is easy to see that when $v(p) = T$, we must have

$$v(A) = T$$

so $\neg$ cannot be expressed using $\vee$ and $\leftrightarrow$ alone.

[5 marks] **(3)** Using the method of semantic tableaux, show that the formula

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

is valid.

Solution:
The tableau is closed.

[5 marks] **(4)** Consider the proof of the valid formula

$$(\neg B \to \neg A) \to ((\neg B \to A) \to B)$$

in the Hilbert's proof system $\mathcal{H}$

Step	Formula	Justification
1.	$\{\neg B \to \neg A, \neg B \to A, \neg B\} \vdash \neg B$	assumption
2.	$\{\neg B \to \neg A, \neg B \to A, \neg B\} \vdash \neg B \to A$	assumption
3.	$\{\neg B \to \neg A, \neg B \to A, \neg B\} \vdash A$	MP 1,2
4.	$\{\neg B \to \neg A, \neg B \to A, \neg B\} \vdash \neg B \to \neg A$	assumption
5.	$\{\neg B \to \neg A, \neg B \to A, \neg B\} \vdash A \to B$	Contrapositive Rule 4
6.	$\{\neg B \to \neg A, \neg B \to A, \neg B\} \vdash B$	MP 3,5
7.	$\{\neg B \to \neg A, \neg B \to A\} \vdash \neg B \to B$	Deduction Rule 6
8.	$\{\neg B \to \neg A, \neg B \to A\} \vdash (\neg B \to B) \to B$	Theorem (class)
9.	$\{\neg B \to \neg A, \neg B \to A\} \vdash B$	MP 7,8
10.	$\{\neg B \to \neg A\} \vdash (\neg B \to A) \to B$	Deduction Rule 9
11.	$\vdash (\neg B \to \neg A) \to ((\neg B \to A) \to B)$	Deduction Rule 10

Provide justification for each step in this proof. You may use any rule or theorem proved or stated in lectures.

[5 marks] **(5)** Convert the following formula into a CNF:

$$(r \rightarrow p) \rightarrow (\neg(q \vee r) \rightarrow p)$$

Solution

$$\begin{aligned}
(r \rightarrow p) \rightarrow (\neg(q \vee r) \rightarrow p) &\equiv \neg(\neg r \vee p) \vee (\neg\neg(q \vee r) \vee p) \\
&\equiv \neg(\neg r \vee p) \vee ((q \vee r) \vee p) \\
&\equiv (r \wedge \neg p) \vee (q \vee r \vee p) \\
&\equiv (r \vee q \vee r \vee p) \wedge (\neg p \vee q \vee r \vee p) \\
&\equiv (p \vee q \vee r) \wedge (\neg p \vee p \vee q \vee r)
\end{aligned}$$

[5 marks] **(6)** Using resolution, determine whether the following set of clauses is satisfiable or not:

$$\{qr\overline{s}, qs, \overline{rs}, \overline{q}\}$$

Solution: Assuming the clauses in the set are enumerated as (1)-(4), we have the following refutation:

$$\begin{array}{lll} 5. & qr & \text{Res } 1,2 \\ 6. & q\overline{s} & \text{Res } 3,5 \\ 7. & q & \text{Res } 2,6 \\ 8. & \square & \text{Res } 4,7 \end{array}$$