Isomorphism

P. Danziger

1 Isomorphism of Graphs

Definition 1 *Given two graphs $G = (V, E)$ and $G' = (V', E')$, we say that they are isomorphic if there exist bijections $f : V \to V'$ and $g : E \to E'$ that preserve the endpoint relations of G and G'. i.e. if $u \in V$ is an endpoint of $e \in E$ then $f(u) \in V'$ is an endpoint of $g(e) \in E'$.*

We write $G \cong G'$.
In order to show that two graphs are isomorphic we must find the bijections f and g.
Notes If G and G' are simple graphs then $G \cong G'$ if and only if there exists a bijection $f : V \to V'$ which preserves the endpoint function.
i.e. $\{u, v\} \in E \Rightarrow \{f(u), f(v)\} \in E'$
Given two graphs G and G' with $G \cong G'$ if G is simple, then so is G'.

Graph Isomorphism Problem Given two graphs G and G' determine whether they are isomorphic.
While there is no known efficient algorithm for the solution of this problem it is not known to be NP-Complete either.
Note if the degree sequence of the vertices of G and G' are different then the Graph Isomorphism Problem is easy.
It is the regular graphs that are bad for the Graph Isomorphism Problem, the less vertices a graph has of the same degree, the easier the Graph Isomorphism Problem becomes.
It can be very easy to show that two graphs are **not** isomorphic by using isomorphic invariants.

Definition 2 *A property P of a graph G is an isomorphic invariant if $G \cong G' \Rightarrow G'$ has property P as well.*

Theorem 3 *The following are all isomorphic invariants of a graph G:*

1. *The number of vertices of G.*
2. *The number of edges of G.*
3. *The number of vertices of a given degree k.*
4. *The degree sequence of G.*
5. *The number of connected components of G.*
6. *G has a circuit of length k.*
7. *The number of simple circuits of length k.*

8. *G has an Euler circuit.*

9. *G has a Hamiltonian circuit.*

Thus if $G \cong G'$ we can say that G and G' have the same number of vertices, edges, degree sequence, etc.

1.1 Isomorphism Classes

We can define a relation on graphs by saying that two graphs are related if and only if they are isomorphic. This is called the graph isomorphism relation.

Theorem 4 *Graph isomorphism is an equivalence relation.*

Proof: Let $G = (V, E)$, $G' = (V', E')$ and $G'' = (V'', E'')$ all be graphs.
Reflexive For all graphs G, $G \cong G$
Take $f = \imath_V$ and $g = \imath_E$.
Symmetric If $G \cong G'$ then $G' \cong G$.
Assume that $G \cong G'$, thus there exists bijections $f : V \to V'$ and $g : E \to E'$ such that if $u \in V$ is an endpoint of $e \in E$ then $f(u) \in V'$ is an endpoint of $g(e) \in E'$.
Since f and g are bijections f^{-1} and g^{-1} are also bijections.
Now, if $f(u) \in V'$ is an endpoint of $g(e) \in E'$ then $f^{-1}(f(u)) = u \in V$ is an endpoint of $g^{-1}(g(e)) \in E$.
Transitive If $G \cong G'$ and $G' \cong G''$ then $G \cong G''$.
Suppose that $G \cong G'$ and $G' \cong G''$, thus there exists bijections $f : V \to V'$, $g : E \to E'$ and $f' : V' \to V''$, $g' : E' \to E''$ which preserve the respective endpoint relations.
Take the bijections $f' \circ f : V \to V''$ and $g' \circ g : E \to E''$.
If $u \in V$ is an endpoint of $e \in E$ then $f(u) \in V'$ is an endpoint of $g(e) \in E'$ and $f'(f(u)) \in V''$ is an endpoint of $g'(g(e)) \in E''$, so these bijections preserve the endpoint relations. $\square$
Since graph isomorphism is an equivalence relation it divides the set of all graphs into equivalence classes.
For most purposes it does not matter which of the isomorphic graphs from an equivalence class we choose.