# Vectors in $\mathbb{R}^n$

P. Danziger

## 1 Vectors

The standard *geometric* definition of vector is as something which has *direction* and *magnitude* but **not** position.
Since vectors have no position we may place them wherever is convenient.
Vectors are often used in Physics to convey information about quantities that have these properties such as velocity and force.
Algebraically, a vector in 2 (real) dimensions is defined to be an ordered pair $(x, y)$, where $x$ and $y$ are both real numbers ($x, y \in \mathbb{R}$).
The set of all 2 dimensional vectors is denoted $\mathbb{R}^2$.

$$\text{i.e. } \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

Algebraically, a vector in 3 (real) dimensions is defined to ba an ordered triple $(x, y, z)$, where $x, y$ and $z$ are all real numbers ($x, y, z \in \mathbb{R}$).
The set of all 3 dimensional vectors is denoted $\mathbb{R}^3$.

$$\text{i.e. } \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

Algebraically, a vector in $n$ (real) dimensions is defined to ba an ordered $n-$tuple $(x_1, x_2, \ldots, x_n)$, where each of the $x_i$ are real numbers ($x_i \in \mathbb{R}$).
The set of all $n$ dimensional vectors is denoted $\mathbb{R}^n$.

$$\text{i.e. } \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ 1 \le i \le n\}$$

A <u>scalar</u> is a number (usually either real or complex).

**Example 1**
$(1, 2) \in \mathbb{R}^2$,
$(1, 2, 3) \in \mathbb{R}^3$,
$(1, 2, 3, 4) \in \mathbb{R}^4$,
$(0, 2, 3, 1, 0, 2, 6) \in \mathbb{R}^7$.
$324 \in \mathbb{R}$ is a scalar.
We use the convention that $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, etc.
If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then the scalars $x_1, x_2, \ldots, x_n$ are called the <u>components</u> of $\mathbf{x}$.

# 2 Points and Vectors

$\mathbb{R}^n$ is defined to be the set of all $n-$tuples, as such they can represent either points or vectors.
We make a distinction between the *points* in $\mathbb{R}^n$ and the vectors. Both are represented by $n-$tuples, but they represent different things.
Points represent static positions in space, points may not be added, nor may they be scalar multiplied.
Points are usually represented by capital letters from the middle of the alphabet, $P, Q, R$, etc.
We generally use lowercase boldface letters from the end of the alphabet ($\mathbf{u}, \mathbf{v}, \mathbf{w} \ldots$) to denote vectors.
Vectors have magnitude and direction, but no position, they may be placed in space wherever is convenient.
Vectors may be added and have scalar multiples taken (see below).
Given any **point** in $P = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ it seems natural to associate it with the **vector** $(x_1, x_2, \ldots, x_n)$ and visa versa.
This is the vector pointing from the origin $(0, 0, \ldots, 0)$ to $P$ and is denoted $\vec{OP}$.
Given two points $P, Q \in \mathbb{R}^n$, where $P = (x_1, x_2, \ldots, x_n)$ and $Q = (y_1, y_2, \ldots, y_n)$ the vector $\vec{PQ}$, whose head reaches $Q$ when its tail is placed at $P$, is formed by the componentwise difference.

$$\vec{PQ} = (y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n) = \vec{OQ} - \vec{OP}$$

Thus, there is a one to one correspondence between the set of vectors and the set of points.
So given two points in 3 space,

$$P = (a, b, c) \text{ and } Q = (d, e, f)$$

the vector joining $P$ to $Q$ is given by

$$\vec{PQ} = (d - a, e - b, f - c) = \vec{OQ} - \vec{OP}.$$

where $O = (0, 0, 0)$ the origin.

**Example 2**
Find the vector joining $P$ to $Q$ where $P = (1, 3, 2)$ and $Q = (1, 1, 4)$.

$$\vec{PQ} = (1 - 1, 1 - 3, 4 - 2) = (0, -2, 2)$$

## 2.1 Scalar Multiplication

Geometrically, given a scalar $k$ and a vector $\mathbf{u} \in \mathbb{R}^n$, $k\mathbf{u}$ is the vector in the same (or opposite) direction as $\mathbf{u}$ but with magnitude $k$ times as large.
If $k > 0$, $k\mathbf{u}$ has the same direction as $\mathbf{u}$.
If $k < 0$, $k\mathbf{u}$ has the opposite direction to $\mathbf{u}$.
Algebraically, given a scalar $k$ and $\mathbf{u} \in \mathbb{R}^n$

$$k\mathbf{u} = k(u_1, u_2, \ldots, u_n) = (ku_1, ku_2, \ldots, ku_n)$$

**Example 3**
$5(1,2,3) = (5,10,15)$

**Definition 4** *Two vectors are* parallel *if they are scalar multiples of each other.*

**Example 5**
Find $a$ such that $(1,2,a)$ is parallel to $(2,4,6)$
Find $a$ such that $(1,2,a)$ is parallel to $(2,4,5)$

## 2.2 Vector Addition

Geometrically, vectors add by placing them head to tail.
Algebraically, given two **vectors**, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we *define* addition componentwise:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \end{aligned}$$

Note that vector addition is only defined if the two vectors are of the same size.
We define subtraction of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$$

**Example 6**
$(1,3) + (1,4) = (2,7)$.
$(-1,3,2) + (1,1,4) = (0,4,6)$.
$(1,0,2,1) + (1,1,2,-2) = (2,1,4,-1)$.
$(2,1,2,-2,0,-1,3) + (1,1,1,1,1,1,1) = (3,2,3,-1,1,0,4)$.
$(1,0,2,1) + (1,1,2)$ is not defined.
$(1,3) - (1,4) = (1,3) + (-1)(1,4) = (1,3) + (-1,-4) = (0,-1)$.
$(2,1,2,-2,0,-1,3) - (1,1,1,1,1,1,1) = (1,0,1,-3,-1,-2,2)$.
Find the point 1/3 of the way from $P = (-1,3,-2)$ to $Q = (2,0,1)$

## 2.3 Special Vectors

The zero vector is a vector, all of whose entries are 0.

$$\mathbf{0} = (0,0,\ldots,0)$$

The zero vector is associated with the origin of the coordinate system.
An elementary vector, $\mathbf{e}_i$ is a vector which has zeros everywhere, except in the $i^{\text{th}}$ position, where it is one.

$$\begin{aligned} \mathbf{e}_1 &= (1,0,\ldots,0) \\ \mathbf{e}_2 &= (0,1,\ldots,0) \\ &\vdots \\ \mathbf{e}_i &= (0,0,\ldots,1,\ldots,0) \qquad \text{1 in } i^{\text{th}} \text{ position} \\ &\vdots \\ \mathbf{e}_n &= (0,0,\ldots,1) \end{aligned}$$

Note that any vector $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ can be written as

$$\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + \ldots + u_n\mathbf{e_n}$$

## 2.4 i, j, k Notation

In $\mathbb{R}^2$ we set use $\mathbf{i}$ to denote the unit vector along the $x-$axis and $\mathbf{j}$ to denote the unit vector along the $y-$axis.

$$\mathbf{i} = e_1 = (1,0), \quad \mathbf{j} = e_2 = (0,1).$$

In $\mathbb{R}^3$ we set use $\mathbf{i}$ to denote the unit vector along the $x-$axis, $\mathbf{j}$ to denote the unit vector along the $y-$axis and $\mathbf{k}$ to denote the unit vector along the $z-$axis

$$\mathbf{i} = e_1 = (1,0,0), \quad \mathbf{j} = e_2 = (0,1,0), \quad \mathbf{k} = e_3 = (0,0,1).$$

Any vector $\mathbf{v} = (a,b) \in \mathbb{R}^2$ can be expressed as

$$a\mathbf{i} + b\mathbf{j}$$

Note that $a\mathbf{i} + b\mathbf{j} = a(1,0) + b(0,1) = (a,b)$.
Any vector $\mathbf{v} = (a,b,c) \in \mathbb{R}^3$ can be expressed as

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Note that

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a(1,0,0) + b(0,1,0) + c(0,0,1) = (a,b,c).$$

## 2.5 Algebraic Properties

**Theorem 7 (Properties of Vectors in $\mathbb{R}^n$)** *Given vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and a scalars $k, \ell \in \mathbb{R}$ then:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ (Existence of Identity)
4. $\mathbf{u} + -\mathbf{u} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ (Existence of Additive Inverse)
5. $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$ (Scalar Associativity)
6. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (Scalar Distributivity I)
7. $(k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$ (Scalar Distributivity II)
8. $1\mathbf{u} = \mathbf{u}$ (Scalar Identity)
9. $0\mathbf{u} = \mathbf{0}$
10. $k\mathbf{0} = \mathbf{0}$

# 3 Dot Product

Given two $n$ dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ we define the vector scalar product or dot product of $\mathbf{u}$ and $\mathbf{v}$ as the sum of the product of the components. So

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (u_1, u_2, \ldots, u_n)\cdot(v_1, v_2, \ldots, v_n) \\ &= u_1v_1 + u_2v_2 + \ldots + u_nv_n.\end{aligned}$$

**Note** that the dot product is defined only for vectors, furthermore the dot product of two vectors yields a scalar.

**Example 8**

$$\begin{aligned}(1,2,3)\cdot(4,5,6) &= 1\times 4 + 2\times 5 + 3\times 6 \\ &= 4 + 10 + 18 \\ &= 32\end{aligned}$$

$$\begin{aligned}(1,2,3,4)\cdot(4,5,6,7) &= 1\times 4 + 2\times 5 + 3\times 6 + 4\times 7 \\ &= 4 + 10 + 18 + 28 \\ &= 60\end{aligned}$$

## 3.1 Properties of dot Product

**Theorem 9** *Given vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and a scalars $k, \ell \in \mathbb{R}$ then:*

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ (Commutativity)
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w} + \mathbf{u}\cdot\mathbf{w}$ (Distributivity)
3. $(k\mathbf{u})\cdot\mathbf{v} = k(\mathbf{v}\cdot\mathbf{u})$ (Associativity)

## 3.2 Magnitude of a Vector

**Definition 10** *The dot product of a vector $\mathbf{u}$ with itself ($\mathbf{u}\cdot\mathbf{u}$) is the square of the length or magnitude of $\mathbf{u}$. We write $||\mathbf{u}|| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$.*

Note In $\mathbb{R}^2$ and $\mathbb{R}^3$

$$||\mathbf{u}|| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \begin{cases}\sqrt{a^2+b^2} & \text{In } \mathbb{R}^2 \\ \sqrt{a^2+b^2+c^2} & \text{In } \mathbb{R}^3\end{cases}$$

**Example 11**
Find the magnitude of the vector $\mathbf{u} = (1,2,3)$

$$\mathbf{u}\cdot\mathbf{u} = (1,2,3)\cdot(1,2,3) = 1 + 4 + 9 = 14$$

Thus $||\mathbf{u}|| = \sqrt{14}$.

**Theorem 12** *Given $\mathbf{u} \in \mathbb{R}^n$, and $k \in \mathbb{R}$:*

1. $||\mathbf{u}|| \geq 0$
2. $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
3. $||k\mathbf{u}|| = |k|\,||\mathbf{u}||$.
4. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (Triangle Inequality).

If $P$ and $Q$ are points in $\mathbb{R}^n$, the distance between $P$ and $Q$ is given by $||\vec{PQ}||$.

**Example 13**

1. Find the distance between $P = (1, 3, 2)$ and $Q = (1, 1, 4)$.

$$||\vec{PQ}|| = ||(0, -2, 2)|| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$$

2. What is the distance from the origin to a point half way between $P = (1, 3, 2)$ and $Q = (1, 1, 4)$.

$$\begin{aligned} ||\vec{OP} + \tfrac{1}{2}\vec{PQ}|| &= ||(1, 3, 2) + \tfrac{1}{2}(0, -2, 2)|| \\ &= ||(1, 1, 4) + \tfrac{1}{2}(0, -2, 2)|| \\ &= \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} \end{aligned}$$

## 3.3 Distance Between Vectors

**Definition 14** *Given two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *the* <u>*distance*</u> *between* $\mathbf{u}$ *and* $\mathbf{v}$*,* $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

**Example 15**
Find the distance between $\mathbf{u} = (1, 3, 2)$ and $\mathbf{v} = (1, 1, 4)$.

$$||\mathbf{v} - \mathbf{u}|| = ||(0, -2, 2)|| = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$$

The distance between two vectors is the same as the distance between their associated points.

**Theorem 16** *Given any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$*:*

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
4. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$

Note that anything which is considered a distance must satisfy these four properties.

## 3.4 Unit Vectors

**Definition 17** *A unit vector is a vector which has unit magnitude, i.e.* $||\mathbf{u}|| = 1$.

**Definition 18** *Given a vector* $\mathbf{v}$ *in* $\mathbb{R}^n$*, the* direction *of* $\mathbf{v}$ *is the unit vector parallel to it.*

Given a vector $\mathbf{v} \in \mathbb{R}^n$, a unit vector parallel to it is given by

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}.$$

Note that $\frac{\mathbf{v}}{||\mathbf{v}||} = \left(\frac{1}{||\mathbf{v}||}\right)\mathbf{v}$

**Example 19**
Find a unit vector parallel to $\mathbf{v} = (1, 1, 1)$.
$||\mathbf{v}|| = \sqrt{1+1+1} = \sqrt{3}$,

$$\text{So } \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{\sqrt{3}}(1, 1, 1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

## 3.5 Unit Vectors in $\mathbb{R}^2$

In $\mathbb{R}^2$ unit vectors are all can be given by

$$(\cos\theta, \sin\theta)$$

where $\theta$ is the angle with the $x-$axis.
Any vector $\mathbf{u}$ in $\mathbb{R}^2$ can be written as

$$||\mathbf{u}||(\cos\theta, \sin\theta)$$

**Example 20**
Find a vector in $\mathbb{R}^2$ which makes an angle of $\frac{\pi}{3}$ with the $x$ axis, and has magnitude 2.

$$\mathbf{u} = 2\left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right) = 2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \left(\sqrt{3}, 1\right)$$

## 3.6 Direction Cosines

Given a vector $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$, the direction of $\mathbf{v}$ is $\frac{\mathbf{v}}{||\mathbf{v}||} = \left(\frac{a}{||\mathbf{v}||}, \frac{b}{||\mathbf{v}||}, \frac{c}{||\mathbf{v}||}\right)$.
The scalars $\alpha = \frac{a}{||\mathbf{v}||}$, $\beta = \frac{b}{||\mathbf{v}||}$ and $\gamma = \frac{c}{||\mathbf{v}||}$ are called the *direction cosines* of $\mathbf{v}$.
They represent the cosines of the angles $\mathbf{v}$ makes with the coordinate axes.

**Example 21**
Find the direction cosines of $\mathbf{v} = \mathbf{j} - \mathbf{k}$.
$||\mathbf{v}|| = \sqrt{2}$,
Thus the cosine of the angle $\mathbf{u}$ makes with the $x-$axis is $\frac{\pi}{2}$. So $\mathbf{u}$ is perpendicular to the $x-$axis.
The cosine of the angle $\mathbf{u}$ makes with the $y-$axis is $\frac{1}{\sqrt{2}}$. So $\mathbf{u}$ makes an angle of $\frac{\pi}{4}$ with the $y-$axis.
The cosine of the angle $\mathbf{u}$ makes with the $z-$axis is $-\frac{1}{\sqrt{2}}$. So $\mathbf{u}$ makes an angle of $\frac{3\pi}{4}$ with the $z-$axis

## 3.7 Angle Between two Vectors

**Theorem 22** *For any two vectors* $\mathbf{u}$ *and* $\mathbf{v}$*,*

$$\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta,$$

*where* $\theta$ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*.*

**Corollary 23** *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if the angle between them is* $\frac{\pi}{2}$*.*

**Example 24**

1. Find the angle between $\mathbf{u} = (1,0,1)$ and $\mathbf{v} = (1,1,0)$

$$\begin{array}{l} \mathbf{u}\cdot\mathbf{v} = (1,0,1)\cdot(1,1,0) = 1+0+0 = 1 \\ ||\mathbf{u}|| = \sqrt{1^2+0^2+1^2} = \sqrt{2} \\ ||\mathbf{v}|| = \sqrt{1^2+1^2+0^2} = \sqrt{2} \\ \therefore \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \\ \text{So } \theta = \frac{\pi}{3} \end{array}$$

2. Find the angle between

$$\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k} \text{ and } \mathbf{v} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}.$$

$$\mathbf{u}\cdot\mathbf{v} = 2+1-1 = 2,\quad ||\mathbf{u}|| = \sqrt{3},\quad ||\mathbf{v}|| = \sqrt{6}.$$

$$\text{So } \cos\theta = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{2}{\sqrt{18}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{18}}\right).$$

## 3.8 Orthogonality

**Definition 25** *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^n$ *are* <u>*orthogonal*</u> *if and only if* $\mathbf{u}\cdot\mathbf{v} = 0$*.*
*A nonempty set is called an* <u>*orthogonal set*</u> *if every pair of vectors from the set is orthogonal.*
*Given a set of vectors in* $S \subseteq \mathbb{R}^n$*, a vector* $\mathbf{u} \in \mathbb{R}^n$ *is* <u>*orthogonal to* $S$</u> *if* $\mathbf{u}$ *is orthogonal to every vector of* $S$*.*

**Example 26**

1. Show that $\mathbf{u} = (1,2,3)$ is orthogonal to $\mathbf{v} = (3,0,-1)$

$$\mathbf{u}\cdot\mathbf{v} = (1,2,3)\cdot(3,0,-1) = 3+0-3 = 0$$

   Since $\mathbf{u}\cdot\mathbf{v} = 0$, $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

2. Find $a$ such that
$$\mathbf{u} = a\mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ is orthogonal to } \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$
When $\mathbf{u}$ is orthogonal to $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} = 0$
$$\text{Now } \mathbf{u} \cdot \mathbf{v} = a + 1 + 2 = a + 3.$$
So $\mathbf{u} \cdot \mathbf{v} = 0$ when $a = -3$.

3. Show that $\{(1, 0, 1, 0), (1, 0, -1, 0), (0, 1, 0, 1)\}$ is an orthogonal set in $\mathbb{R}^4$
$$\begin{aligned} (1,0,1,0) \cdot (1,0,-1,0) = 1 - 1 = 0 \\ (1,0,1,0) \cdot (0,1,0,1) = 0 \\ (1,0,-1,0) \cdot (0,1,0,1) = 0 \end{aligned}$$

4. Let $S = \{(x, y, z) \mid 2x + y - 3z = 0\}$. Show that $\mathbf{u} = (2, 1, -3)$ is orthogonal to $S$.
Let $\mathbf{x} = (x, y, z)$ be an arbitrary vector in $S$, so $2x + y - 3z = 0$
Consider $\mathbf{u} \cdot \mathbf{x} = (2, 1, -3) \cdot (x, y, z) = 2x + y - 3z = 0$
Since $\mathbf{x}$ was chosen arbitrarily from $S$, $\mathbf{u}$ is orthogonal to every vector in $S$.

**Theorem 27 (Generalised Pythagoras' Theorem)** *Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ which are orthogonal ($\mathbf{u} \cdot \mathbf{v} = 0$) then $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.*

**Proof:**
$$\begin{aligned} ||\mathbf{u} + \mathbf{v}|| &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \end{aligned}$$

**Theorem 28 (Triangle Inequality)** *Given any two vectors $\mathbf{u}$ and $\mathbf{v}$*
$$||\mathbf{u} + \mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$$