Linear Combinat$^{\wedge}$

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eg Let $\underline{v}_1 = (1,1,1)$, $\underline{v}_2 = (1,0,1)$, $\underline{v} = (3,1,3)$

Expres $\underline{v}$ as a lin. comb. of $\underline{v}_1$ & $\underline{v}_2$

Lin. comb. $\underline{v} = a_1 \underline{v}_1 + a_2 \underline{v}_2$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{array}\right)$$

$$a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\3\end{pmatrix}$$

$$a_1 + a_2 = 3$$
$$a_1 + \quad = 1$$
$$a_1 + a_2 = 3$$

$a_1 = 1, \ a_2 = 2$

$\underline{v} = \underline{v}_1 + 2\underline{v}_2$

Span

Find the span of $\underline{v}_1$ & $\underline{v}_2$

eg i) What is $\text{sp}\{ \overset{v_1}{(1,1,1)}, \overset{v_2}{(1,0,1)} \}$

Consider $\underline{x} = \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$ [Want all $\underline{x} \in \mathbb{R}^3$ s.t. $\underline{x} = a_1\underline{v}_1 + a_2\underline{v}_2$]

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 1 & 0 & y \\ 1 & 1 & z \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_2 - R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & -1 & y-x \\ 0 & 0 & z-x \end{array}\right)$$

3rd row says Sol^{n}'s only if $x - z = 0$

So $\text{sp}\{\underline{v}_1, \underline{v}_2\} = \{\underline{x} \in \mathbb{R}^3 \mid x - z = 0\}$ $(\underline{x} = (x,y,z)$

i.e. $\underline{v}_1$ & $\underline{v}_2$ span the plane $x - z = 0$.

[Note $\underline{n} = (1,0,-1)$ $\underline{n}\cdot\underline{v}_1 = (1,0,-1)\cdot(1,1,1) = 0$
& $\underline{n}\cdot\underline{v}_2 = (1,0,-1)\cdot(1,0,1) = 0$
So $\underline{v}_1$ & $\underline{v}_2$ lie in the plane]

2) Show that $\underline{v}_1 = (1,1,1)$ $\underline{v}_2 = (1,0,1)$ $\underline{v}_3 = (0,1,1)$ span $\mathbb{R}^3$

[Must show that any vector in $\mathbb{R}^3$ can be expressed as a linear comb. of these 3]

Let $\underline{x} \in \mathbb{R}^3$ Find a_1, a_2, a_3 s.t. $\underline{x} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3$

$$a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\1\end{pmatrix} + a_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & x\\1 & 0 & 1 & y\\1 & 1 & 1 & z\end{array}\right) \quad \begin{array}{l}R_2 \to R_2 - R_1\\R_3 \to R_3 - R_1\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & x\\0 & -1 & 1 & y-x\\0 & 0 & 1 & z-x\end{array}\right)$$

– Unique sol^{n} for each $(x,y,z) \in \mathbb{R}^3$ <u>so spans</u>

b) Express $\underline{u} = (\overset{x}{3}, \overset{y}{1}, \overset{z}{4})$ in terms of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & x\\0 & -1 & 1 & y-x\\0 & 0 & 1 & z-x\end{array}\right) \quad R_2 \to R$$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & x\\0 & 1 & -1 & x-y\\0 & 0 & 1 & z-x\end{array}\right) \quad R_2 \to R_2 + R_3$$

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & x\\0 & 1 & 0 & x-y+z-x\\0 & 0 & 1 & z-x\end{array}\right) = z-y \quad R_1 \to R_1 - R_2$$

$x=3, y=1, z=4$

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & x-z+y\\0 & 1 & 0 & z-y\\0 & 0 & 1 & z-x\end{array}\right) \quad \text{So} \quad \begin{array}{ll}a_1 = x-z+y & = 0\\a_2 = z-y & = 3\\a_3 = z-x & = 1\end{array}$$

$\therefore \quad u = 3\underline{v}_2 + v_3$ Could write u = (

Could write $\underline{u} = (0, 3, 1)_{\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}}$

Where these are the coords of $\underline{u}$ w.r.t. the 'basis' $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

3) Find span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ where

$\underline{v}_1 = (1, 0, 1)$, $\underline{v}_2 = (0, 1, 1)$ $\underline{v}_3 = (1, 1, 2)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y \\ 1 & 1 & 2 & z \end{array}\right) \quad R_3 \to R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y \\ 0 & 1 & 1 & z - x \end{array}\right) \quad R_3 \to R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 0 & z - x - y \end{array}\right)$$

3rd row says $0 = z - x - y$

So span of $\underline{v}_1, \underline{v}_2, \underline{v}_3$ is the plane $x + y - z = 0$

Note $\underline{v}_1, \underline{v}_2\ \underline{v}_3$ Do not span $\mathbb{R}^3$. They all lie in the plane.

b) Express $(1, 0, 2)$ as a lin. comb. of $\underline{v}_i$

$$1 + 0 - 2 = -1 \neq 0$$

$(1, 0, 2)$ Does not lie in the plane so cannot be expressed as a combinat^{n} of $\underline{v}_i$

3) Find $sp\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ where $\underline{v}_1, \underline{v}_2$ & $\underline{v}_3$ are as above. & $\underline{v}_4 = (0,0,1)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 1 & 1 & 2 & 1 & z \end{array}\right) \quad R_3 \to R_3 - R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 1 & 1 & 1 & z-x \end{array}\right) \quad R_3 \to R_3 - R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 0 & 1 & z-x-y \end{array}\right)$$

Σ Multiple sol$^{\underline{n}}$s for any x, y, z.

So span of $\underline{v}_1 - \underline{v}_4$ is $\mathbb{R}^3$

Actually $\underline{v}_3 = \underline{v}_1 + \underline{v}_2$

So given any sol$^{\underline{n}}$ to $\underline{u} = a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 + a_4\underline{v}_4$

we can replace $\underline{v}_3$ with $\underline{v}_1 + \underline{v}_2$ to get

$$\underline{u} = a_1 v_1 + a_2 \underline{v}_2 + a_3(\underline{v}_1 + \underline{v}_2) + a_4 \underline{v}_4$$
$$= (a_1 + a_3)\underline{v}_1 + (a_2 + a_3)\underline{v}_2 + a_4\underline{v}_4$$

Don't need $\underline{v}_3$.

$c_1 = 1, c_2 = 1, c_3 = -1, c_4 = 0$

Since $\underline{v}_3 = \underline{v}_1 + \underline{v}_2 \Rightarrow \underline{v}_1 + \underline{v}_2 - \underline{v}_3 = \underline{0}$

$\Rightarrow \underline{v}_1, \underline{v}_2 - \underline{v}_4$ are lin. dependent.

Any time a set of vectors is lin. dep. we can find a relat$^{\underline{n}}$ between them.