Euclidean n Space

P. Danziger

1 Euclidean n Space

1.1 Definitions

Definition 1

1. An ordered n-tuple is an ordered sequence of n real numbers $(x_1, x_2, \ldots, x_n)$.
   If $n = 2$ we have an ordered pair.
   If $n = 3$ we have an ordered triple.

   n-tuples can either represent points or vectors.

   We use the convention that $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, etc.

2. The set of all possible n-tuples for a fixed n is denoted $\mathbb{R}^n$.
   ie $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R} \text{ for each } i\}$.

3. Given two **vectors**, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we *define* addition termwise:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \end{aligned}$$

4. Given a vector $\mathbf{u} \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$, we define scalar multiplication termwise:

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n).$$

5. The zero vector is a vector, all of whose entries are 0.

$$\mathbf{0} = (0, 0, \ldots, 0)$$

6. An elementary vector, $\mathbf{e}_i$ is a vector which has zeros everywhere, except in the i^{th} position, where it is one.

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \ldots, 0) \\ \mathbf{e}_2 &= (0, 1, \ldots, 0) \\ &\vdots \\ \mathbf{e}_i &= (0, 0, \ldots, 1, \ldots, 0) \qquad \text{1 in } i^{\text{th}} \text{ position} \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \ldots, 1) \end{aligned}$$

Theorem 2 (Properties of Vectors in $\mathbb{R}^n$) *Given vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and a scalars* $k, \ell \in \mathbb{R}$ *then:*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutativity)
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (Associativity)
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ (Existence of Identity)
4. $\mathbf{u} + -\mathbf{u} = -\mathbf{u} + \mathbf{u} = \mathbf{0}$ (Existence of Additive Inverse)
5. $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$ (Scalar Associativity)
6. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (Scalar Distributivity I)
7. $(k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$ (Scalar Distributivity II)
8. $1\mathbf{u} = \mathbf{u}$

1.2 Vector Scalar Product or Dot Product

Given two n dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ we define the vector scalar product or dot product of $\mathbf{u}$ and $\mathbf{v}$ as the sum of the product of the components. So

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (u_1, u_2, \ldots, u_n) \cdot (v_1, v_2, \ldots, v_n) \\ &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.\end{aligned}$$

Note that the dot product is defined only for vectors, furthermore the dot product of two vectors yields a scalar.

Example 3

$$(1, 2, 3) \cdot (4, 5, 6) = 1 \times 4 + 2 \times 5 + 3 \times 6 = 4 + 10 + 18 = 32$$

Theorem 4 *Given vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and a scalars* $k, \ell \in \mathbb{R}$ *then:*

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Commutativity)
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} + \mathbf{u} \cdot \mathbf{w}$ (Distributivity)
3. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{v} \cdot \mathbf{u})$ (Associativity)

Theorem 5 *For any vector* $\mathbf{v} \in \mathbb{R}^n$ $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$

1.3 Length and the Distance between two Vectors

Definition 6 *The dot product of a vector* $\mathbf{u}$ *with itself* ($\mathbf{u} \cdot \mathbf{u}$) *is the square of the* <u>*length*</u> *or* <u>*magnitude*</u> *of* $\mathbf{u}$. *We write* $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

Theorem 7 *Given* $\mathbf{u} \in \mathbb{R}^n$, *and* $k \in \mathbb{R}$:

1. $||\mathbf{u}|| \geq 0$
2. $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
3. $||k\mathbf{u}|| = |k| \, ||\mathbf{u}||$.
4. $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (Triangle Inequality).

Example 8
Find the magnitude of the vector $\mathbf{u} = (1, 2, 3)$

$$\mathbf{u} \cdot \mathbf{u} = (1, 2, 3) \cdot (1, 2, 3) = 1 + 2 + 9 = 14$$

Thus $||\mathbf{u}|| = \sqrt{14}$.

Definition 9 *Given two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *the* <u>*distance*</u> *between* $\mathbf{u}$ *and* $\mathbf{v}$, $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

Theorem 10 *Given any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$:

1. $d(\mathbf{u}, \mathbf{v}) \geq 0$
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
4. $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$

Note that anything which is considered a distance must satisfy these four properties.

1.4 Orthogonality

Definition 11 *Two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *are* <u>*orthogonal*</u> *if and only if* $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 12
Show that $\mathbf{u} = (1, 2, 3)$ is orthogonal to $\mathbf{v} = (3, 0, -1)$

$$\mathbf{u} \cdot \mathbf{v} = (1, 2, 3) \cdot (3, 0, -1) = 3 + 0 - 3 = 0$$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

Theorem 13 (Generalised Pythagoras' Theorem) *Given two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *which are orthogonal* ($\mathbf{u} \cdot \mathbf{v} = 0$) *then* $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Proof:

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}|| &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \end{aligned}$$

1.5 The Angle Between two Vectors

Theorem 14 *Given two vectors* $\mathbf{u}$ *and* $\mathbf{v}$

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$$

where θ *is the angle between the two vectors.*

Corollary 15 *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if the angle between them is* $\frac{\pi}{2}$*.*

Example 16

Find the angle between $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (1, 1, 0)$

$$\begin{array}{l}
\mathbf{u} \cdot \mathbf{v} = (1, 0, 1) \cdot (1, 1, 0) = 1 + 0 + 0 = 1 \\
||\mathbf{u}|| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2} \\
||\mathbf{v}|| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2} \\
\therefore \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \\
\text{So } \theta = \frac{\pi}{3}
\end{array}$$