Large Families of Mutually Embeddable Vertex-Transitive Graphs

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Abstract: For each infinite cardinal κ, we give examples of \(2^\kappa\) many non-isomorphic vertex-transitive graphs of order κ that are pairwise isomorphic to induced subgraphs of each other. We consider examples of graphs with these properties that are also universal, in the sense that they embed all graphs with smaller orders as induced subgraphs. © 2003 Wiley Periodicals, Inc.


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1. INTRODUCTION

All the graphs we consider are undirected and simple. We use the notation of [6] for graph theory, and [10] for basic facts on ordinals and cardinals. We work within ZFC; no additional set-theoretic axioms will be assumed unless stated otherwise.

The countably infinite random graph $R$ is the unique countable graph, up to isomorphism, satisfying the existentially closed or e.c. property: for each pair of finite disjoint sets $S$ and $T$ of vertices, there is a vertex $z \notin S \cup T$ adjacent to each vertex of $S$, and not adjacent to any vertex of $T$. Furthermore, $R$ is homogeneous: every isomorphism between finite induced subgraphs extends to an automorphism. The proof of these results uses the back and forth method, where a bijection and its inverse are constructed recursively at the same time; see [3]. To a neophyte, this is perhaps surprising: one’s initial instinct is to first construct a function and then prove that it is bijective. The Cantor-Schröder-Bernstein theorem (see [1]) proves that mutual embeddings of sets imply the existence of a bijection between the sets. The analogous property is false for graphs, where the role of bijections is played by graph isomorphisms. We write $G \cong H$ if $G$ is isomorphic to an induced subgraph of $H$; we then say that $G$ embeds in $H$. We write $G \sim H$ if $G \leq H$ and $H \leq G$, and we write $G \cong H$ if $G$ is isomorphic to $H$. A family $\{G_i\}_{i \in I}$ of graphs is called mutually embeddable if for every $i, j \in I$, $G_i \cong G_j$.

There exist many families of graphs containing non-isomorphic mutually embeddable members. For example, let $X$ be the set of all infinite subsets of positive integers. Given $s \in X$, define $G_s$ to be the disjoint union of each of the graphs $P_{s_i}$, where $s = \{s_i : i \in \mathbb{N}\}$. Then for distinct $s$ and $s'$ in $X$, $G_s \sim G_{s'}$ but $G_s \not\cong G_{s'}$. Hence, $G_X = \{G_s : s \in X\}$ is a mutually embeddable family of countable graphs, and $G_X$ contains $2^{|\mathbb{N}|}$ many non-isomorphic graphs.

In the present article, we consider an open-ended question motivated in part by the examples in the previous paragraph: which graph properties guarantee that the existence of mutual embeddings implies the existence of an isomorphism? At the other extreme, we may also ask how many properties can graphs in a family share with $R$ and still contain uncountable families of mutually embeddable graphs. For instance note that $R$ is vertex-transitive while none of the graphs $G_s$ of the previous paragraph are. Does there also exist examples of mutually embeddable non-isomorphic vertex-transitive graphs? We note that mutually embeddable locally finite vertex-transitive graphs are necessarily isomorphic. Indeed, if $G$ and $H$ have these properties, then $G$ and $H$ are regular, and the existence of mutual embeddings implies that they must have the same degree. Hence, an embedding of $G$ into $H$ acts locally as an isomorphism, and so it maps the connected components of $G$ to isomorphic connected components of $H$. The Cantor-Schröder-Bernstein theorem then implies that $G$ and $H$ are isomorphic.
We will show that in contrast, for every infinite cardinal $\kappa$ there are $2^\kappa$ many vertex-transitive mutually embeddable non-isomorphic graphs of order $\kappa$; see Theorems 3.1 and 3.2. Note that for an infinite cardinal $\kappa$, there are $2^\kappa$ many non-isomorphic graphs of order $\kappa$, so Theorems 3.1 and 3.2 are the best possible results in this direction. In general it is not a simple matter to prove that two graphs are not isomorphic. Our construction uses the weak Cartesian product of graphs (see [6]), and we use the unique factorization property of the product to show that the graphs are indeed non-isomorphic. To the authors knowledge, this is the first application of the unique factorization theorem for weak Cartesian products with infinitely many factors. In Theorems 4.1 and Corollary 4.1 we strengthen Theorem 3.1 to include graphs $G$ that have the additional property, shared by $R$, of universality: $G$ embeds each graph of order at most the order of $G$. However, our proofs of these results use the Generalized Continuum Hypothesis in the uncountable case.

2. THE WEAK CARTESIAN PRODUCT OF GRAPHS

The Cartesian product of a family $\{G_i\}_{i \in I}$ of graphs is the graph $\prod_{i \in I} G_i$ defined by

$$V\left(\prod_{i \in I} G_i\right) = \left\{ f : I \rightarrow \bigcup_{i \in I} V(G_i) : f(i) \in V(G_i) \text{ for all } i \in I \right\},$$

$$E\left(\prod_{i \in I} G_i\right) = \{fg : \text{there exists } j \in I \text{ such that } f(j)g(j) \in E(G_j) \text{ and } f(i) = g(i) \text{ for all } i \neq j\}.$$

If $I$ is infinite and all the factors $G_i$ have at least two vertices, then $\prod_{i \in I} G_i$ is necessarily disconnected. The weak Cartesian product is an induced subgraph of the Cartesian product where vertices can differ in only finitely many coordinates. For $f \in V(\prod_{i \in I} G_i)$, the weak Cartesian product of $\{G_i\}_{i \in I}$ with base $f$ is the subgraph $\prod'_{i \in I} G_i$ of $\prod_{i \in I} G_i$ induced by the functions $g \in V(\prod_{i \in I} G_i)$ such that $\{i \in I : g(i) \neq f(i)\}$ is finite.

If the graphs $G_i, i \in I$, are connected, then $\prod'_{i \in I} G_i$ is the connected component of $\prod_{i \in I} G_i$ containing $f$. Hence, a weak Cartesian product of connected graphs is necessarily connected. If for some infinite cardinal $\kappa$ we have $|V(G_i)| \leq \kappa$ for all $i \in I$ and $|I| \leq \kappa$, then $|V(\prod'_{i \in I} G_i)| \leq \kappa$.

A graph $G$ is called prime if it cannot be expressed as a weak Cartesian product of graphs in a non-trivial way. The unique factorization properties of connected graphs under the weak Cartesian product can be summarized as follows.

**Theorem 2.1** (see Theorem B.5 of [6]). Every connected graph admits a unique representation as a weak Cartesian product of prime graphs. Furthermore for any
isomorphism $\phi$ between $G = \prod_{i \in I} G_i, H = \prod_{j \in J} H_j$, where all factors are connected and prime, there exists a bijection $\psi : J \rightarrow I$ and isomorphisms $\psi_j : G_{\psi(j)} \rightarrow H_j$, for $j \in J$ such that for all $f \in V(G)$,

$$\phi(f)(j) = \psi_j(f(\psi(j))).$$

We will also use the following:

**Remark 1.** The weak Cartesian product of connected vertex-transitive graphs is vertex-transitive.

This result is not hard to deduce from the fact that automorphisms of the factors applied coordinatewise yield an automorphism of the product. Sabidussi [11], Miller [7], and Imrich [4] give more information on the structure of the automorphism group of the Cartesian product of graphs. Imrich [5] was the first to note that the converse does not hold: there exist vertex-transitive connected graphs whose prime factors are not vertex-transitive. The construction is adapted to a family of suitable graphs in the next section.

### 3. THE CONSTRUCTION

Throughout, $\kappa$ will be a fixed infinite cardinal, identified with the set $\{\alpha < \kappa : \alpha$ is an ordinal$\}$. We first define by transfinite induction a family of rooted trees $\{T_\alpha\}_{\alpha \in \kappa}$, as follows:

(i) $T_0$ is the one-vertex tree with root $u_0$.

(ii) $T_{\alpha+1}$ is obtained from two disjoint copies of $T_\alpha$ by adding a new root $u_{\alpha+1}$ adjacent to the two copies of the root $u_\alpha$ of $T_\alpha$.

(iii) If $\alpha$ is a limit ordinal, then $T_\alpha$ is obtained from the disjoint union of $\{T_\beta\}_{\beta < \alpha}$ by adding a new root $u_\alpha$ adjacent to the copies of the root $u_\beta$ of $T_\beta$ for all $\beta < \alpha$.

The following lemma has a simple proof, but it is fundamental to our approach.

**Lemma 3.1.** For ordinals $0 \leq \alpha < \beta < \kappa$, $T_\alpha \leq T_\beta$, but $T_\alpha \not\cong T_\beta$.

**Proof.** The ordinal $\alpha$ is an isomorphism invariant of $T_\alpha$: it is the (ordinal) number of times we need to repeat the operation of removing all vertices of degree 1 in order to reduce $T_\alpha$ to a single vertex. Hence, $T_\alpha \not\cong T_\beta$. Suppose that $T_\alpha \not\cong T_\beta$. Then there exists a least ordinal $\gamma \leq \beta$ such that $T_\alpha \not\cong T_\gamma$. However, by the definition of $T_\gamma$, we have $T_\alpha \not\cong T_\gamma$ if $\gamma$ is a limit ordinal, and $T_6 \not\cong T_\gamma$ if $\gamma = \delta + 1$. Hence $T_\alpha \not\cong T_\gamma$ implies $T_\alpha \not\cong T_6$, which contradicts the minimality of $\gamma$. \qed

We now use the weak Cartesian product to construct a family $\{G_\alpha\}_{\alpha < \kappa}$ of vertex-transitive graphs with the properties of the $T_\alpha$ described in Lemma 3.1.
Note that all trees are prime, since a Cartesian product of non-trivial graphs necessarily contains 4-cycles; hence, all the graphs $T_\alpha$ are prime. Following [6], we construct vertex-transitive weak Cartesian products whose prime factors are all isomorphic to the same $T_\alpha$.

For $\alpha \in \kappa$, let $I_\alpha = \kappa \times V(T_\alpha)$, and define $f_\alpha : I_\alpha \to V(T_\alpha)$ by $f_\alpha(\beta, v) = v$. Let $G_\alpha = \prod_{\beta \in I_\alpha} T_\alpha$. Note that $|V(T_\alpha)| \leq \kappa$, hence $|V(G_\alpha)| \leq \kappa$.

**Lemma 3.2.** The graphs $G_\alpha$, where $\alpha \in \kappa$, are vertex-transitive, and for ordinals $0 \leq \alpha < \beta < \kappa$, $G_\alpha \leq G_\beta$, but $G_\alpha \not\cong G_\beta$.

**Proof.** For every $g \in V(G_\alpha)$, $v \in V(T_\alpha)$, we have that

$$|g^{-1}(v)| = |f_\alpha^{-1}(v)| = \kappa.$$ 

Therefore, there exists a bijection from $g^{-1}(v)$ to $f_\alpha^{-1}(v)$. Fixing such a bijection for each $v \in V(T_\alpha)$, we get a bijection $\phi$ from the set $I_\alpha$ to itself, such that for every $(\gamma, v) \in I_\alpha$, $g(\gamma, v) = f_\alpha(\phi(\gamma, v))$. Let $\psi : V(G_\alpha) \to V(G_\alpha)$ be the function defined by $\psi(f) = f'$, where $f'(\gamma, v) = f(\phi(\gamma, v))$. Then $\psi$ is an automorphism of $G_\alpha$, with $\psi(f_\alpha) = g$. This shows that $G_\alpha$ is vertex-transitive.

By Lemma 3.1, for $\alpha < \beta$, there exists an embedding $g_{\alpha, \beta}$ of $T_\alpha$ into $T_\beta$. Define a function $F_{\alpha, \beta} : g_{\alpha, \beta} \to G_\beta$ by $F_{\alpha, \beta}(f) = \hat{f}$, where

$$\hat{f}(\gamma, v) = \begin{cases} g_{\alpha, \beta} \circ f(\gamma, v^{-1}(v)) & \text{if } v \in g_{\alpha, \beta}(T_\alpha); \\ v & \text{otherwise.} \end{cases}$$

To see that $F_{\alpha, \beta}$ is an embedding, suppose that $f$ and $g$ are adjacent in $G_\alpha$. Hence, there is a $(\gamma, v) \in I_\alpha$ so that $f(\gamma, v)g(\gamma, v) \in E(T_\alpha)$, and $f(j) = g(j)$ for all $j \in I_\alpha \setminus \{\gamma, v\}$. Then $\hat{f}(\gamma, g_{\alpha, \beta}(v)) = f(\gamma, v)$ is adjacent to $\hat{g}(\gamma, g_{\alpha, \beta}(v)) = g(\gamma, v)$. As $\hat{f}$ and $\hat{g}$ are equal on all $j \in I_\alpha \setminus \{\gamma, g_{\alpha, \beta}(v)\}$, they are adjacent in $G_\beta$. The verification that $F_{\alpha, \beta}$ is injective and preserves non-edges is similar, using the facts that each $g_{\alpha, \beta}$ is an embedding.

The prime factors of $G_\alpha$ are isomorphic to $T_\alpha$ and all those of $G_\beta$ to $T_\beta$. Hence, by Theorem 2.1 and Lemma 3.1, we have that $G_\alpha \not\cong G_\beta$.

Note that $T_0 \cong G_0$ is the graph with one vertex, which is the unit for the Cartesian product. We will discard this graph and base our construction on the other graphs in the family $\{G_\alpha : \alpha < \kappa\}$.

Let

$$X_\kappa = \{I \subseteq \kappa \setminus \{0\} : |I| = \kappa\}.$$ 

Then $|X_\kappa| = 2^\kappa$. We say that a set of ordinals $S$ is cofinal in $\kappa$ if for each ordinal $\gamma < \kappa$, there exists $\alpha \in S$, so that $\gamma < \alpha$. Note that each $I \in X_\kappa$ is cofinal in $\kappa$. If not, then there is an ordinal $\gamma < \kappa$ so that all $\alpha \in I$ satisfy $\alpha \leq \gamma$. But then $|I| \leq |\gamma| < \kappa$, which is a contradiction.
For \( I \in X_\kappa \), we define \( H_I = \prod_{\alpha \in I}^\kappa G_\alpha \), where \( f_I : I \rightarrow \cup_{\alpha \in I} V(G_\alpha) \) is any function such that \( f_I(\alpha) \in V(G_\alpha) \) for all \( \alpha \in I \). (The vertex-transitivity of \( G_\alpha \) makes the choice of \( f_I(\alpha) \) irrelevant.) Since \( |I| = \kappa \), we have that \( |V(H_I)| = \kappa \).

**Lemma 3.3.** The family \( \{H_I\}_{I \in X_\kappa} \) is mutually embeddable and consists of vertex-transitive non-isomorphic graphs.

**Proof.** The graphs \( H_I \) are all vertex-transitive by Remark 1. For \( I, J \in X_\kappa \), we can define an injection \( \phi : I \rightarrow J \) such that \( \phi(\alpha) > \alpha \) for all \( \alpha \in I \). We define \( \phi \) by transfinite induction. If \( S \) is a nonempty set of ordinals, then let \( \min(S) \) be the least ordinal in \( S \).

(i) If \( \alpha_0 = \min(I) \), then let \( \phi(\alpha_0) = \min(J \setminus \{\alpha_0\}) \). Observe that \( J \setminus \{\alpha_0\} \neq \emptyset \) since \( J \) is cofinal in \( \kappa \).

(ii) If \( \alpha + 1 \in I \), then define

\[
\phi(\alpha + 1) = \min(J \setminus (\{\alpha + 1\} \cup \{\phi(\beta) : \beta \leq \alpha \text{ and } \beta \in I\})).
\]

(iii) If \( \alpha \) is a limit ordinal in \( I \), then

\[
\phi(\alpha) = \min(J \setminus (\alpha \cup \{\phi(\beta) : \beta < \alpha \text{ and } \beta \in I\})).
\]

For all \( \alpha \in I \), we can select, by Lemma 3.2 and since \( G_{\phi(\alpha)} \) is vertex-transitive, an embedding \( \psi_\alpha : G_\alpha \rightarrow G_{\phi(\alpha)} \) such that \( \psi_\alpha(f_I(\alpha)) = f_J(\phi(\alpha)) \). The embedding \( \psi : H_I \rightarrow H_J \) is then defined by \( \psi(f) = f' \), where

\[
f'(j) = \begin{cases} 
\psi_{\phi^{-1}(j)}(f(\phi^{-1}(j))) & \text{if } j \in \phi(I); \\
\psi_j(j) & \text{otherwise}.
\end{cases}
\]

Observe that \( \psi(f_I) = f_J \). To show that \( \psi \) is an embedding, consider an edge \( fg \in E(H_I) \). Then there is a \( j \in I \) so that \( f(j)g(j) \in E(G_J) \) and \( f(i) = g(i) \) for all \( i \in I \setminus \{j\} \). Then \( f'(\psi(j)) = \psi_{\phi(j)}(f(j)) \) is adjacent to \( g'(\psi(j)) = \psi_{\phi(j)}(g(j)) \) in \( G_{\phi(j)} \), since \( \psi_{\phi(j)} \) is an embedding. As \( f' \) and \( g' \) are equal on all \( i \in I \setminus \{\psi(j)\} \), they are adjacent in \( H_J \). The verification that \( \psi \) is injective and preserves edges is similar, and so is omitted.

Thus, the graphs \( \{H_I\}_{I \in X_\kappa} \) are mutually embeddable. However, for \( I \neq J \), the prime Cartesian factors of \( H_I \) are those of \( \{G_\alpha\}_{\alpha \in I} \), namely the trees \( \{T_\alpha\}_{\alpha \in I} \), while those of \( H_J \) are the trees \( \{T_\alpha\}_{\alpha \in J} \). Lemma 3.1 shows that \( H_I \) and \( H_J \) are not isomorphic.

Our main theorem is the following.

**Theorem 3.1.** For every infinite cardinal \( \kappa \), there exist \( 2^\kappa \) many order \( \kappa \), vertex-transitive, mutually embeddable, non-isomorphic connected graphs.

**Proof.** Apply Lemma 3.3 along with the fact that \( |X_\kappa| = 2^\kappa \).
Several aspects of our construction can easily be modified. For instance, for every connected prime graph $T$ and any infinite cardinal $\kappa \geq |V(T)|$, there exists a vertex-transitive graph $G(T, \kappa)$ which is a weak Cartesian product of copies of $T$ and such that $|V(G(T, \kappa))| = \kappa$. The construction of $G(T, \kappa)$ parallels that of $G_\alpha$ given above. The chain $\{T_\alpha\}_{\alpha \leq \kappa}$ of graphs may also be replaced by certain other chains. For example, in step (ii) of the definition of the trees $T_\alpha$, we may replace 2 by 3, or by $n$, if $n \geq 4$. If $\kappa$ is a cardinal, then we say that a class $C$ of graphs is a $\kappa$-chain if the following conditions hold.

1. If $G \in C$, then $|V(G)| \leq \kappa$.
2. For each ordinal $\alpha \in \kappa \setminus \{0\}$, there is a unique graph $J_\alpha \in C$.
3. If $0 \leq \alpha < \beta \in \kappa$, then $J_\alpha \leq J_\beta$, but $J_\alpha \not\cong J_\beta$.
4. The graphs in $C$ are prime and connected.

By replacing the role of the trees $T_\alpha$ in Lemmas 3.2 and 3.3 by graphs in a $\kappa$-chain, we obtain the following strengthening of Theorem 3.1.

**Theorem 3.2.** Let $\kappa$ be a fixed infinite cardinal, and let $C$ be a fixed $\kappa$-chain. There exists $2^\kappa$ many order $\kappa$, vertex-transitive, mutually embeddable, non-isomorphic connected graphs, each of whose prime factors are in $C$.

**4. FAMILIES OF UNIVERSAL VERTEX-TRANSITIVE GRAPHS**

A graph of order $\kappa$ is $\kappa$-universal if it embeds all graphs of order at most $\kappa$ as induced subgraphs. For example, the random graph $R$ is $\aleph_0$-universal. This was first proved by Rado [8,9], who also proved that if the Generalized Continuum Hypothesis (GCH) holds, then $\kappa$-universal graphs exist for all $\kappa > \aleph_0$. (The (GCH) says that for every cardinal $\alpha$, $\aleph_{\alpha+1} = 2^\alpha$.) Every family of $\kappa$-universal graphs is mutually embeddable. One question is whether there is a family of cardinality $2^\kappa$ consisting of non-isomorphic vertex-transitive graphs that are $\kappa$-universal. If so, then we say that such a family is $\kappa$-good.

The Cartesian product of two graphs $G$ and $H$ is written $G \square H$.

**Theorem 4.1.** Let $\kappa \geq \aleph_0$. If there is a $\kappa$-universal graph, then there is a $\kappa$-good family.

**Proof.** Fix $\kappa \geq \aleph_0$ and let $U$ be a $\kappa$-universal graph. Let $T$ be the graph consisting of $U$ and a vertex $u$ adjacent to each vertex of $U$. Then $T$ is a connected $\kappa$-universal graph. Furthermore, $T$ is prime: the new vertex $u$ is adjacent to every other vertex, therefore it is not contained in any induced 4-cycle. However every vertex of a non-trivial weak Cartesian product of graphs (with no isolated vertices) is contained in an induced 4-cycle.

As noted below the proof of Theorem 3.1 there exists a vertex-transitive graph $G(T, \kappa)$ of order $\kappa$ which is a weak Cartesian product of copies of $T$; in particular, $G(T, \kappa)$ contains a copy of $U$. For a fixed $I \in X_\kappa$, define $U_I = G(T, \kappa) \square H_I$. By Remark 1, $U_I$ is a $\kappa$-universal vertex-transitive graph whose prime factors consist
of $T$ and the trees $T_\alpha$, where $\alpha \in I$. Since $T$ contains cycles, $T \not\cong T_\alpha$ for all $\alpha$. The fact that there are $2^\kappa$ many non-isomorphic graphs of the form $U_I$ now follows by Theorem 2.1.

**Corollary 4.1.** 1. There is an $\aleph_0$-good family.

2. Assuming (GCH), there is a $\kappa$-good family for each $\kappa > \aleph_0$.

**Proof.** For item (1), apply Theorem 4.1 using the infinite random graph $R$. For item (2), by [8,9] and by assuming (GCH), there is a $\kappa$-universal graph for each cardinal $\kappa > \aleph_0$. Now apply Theorem 4.1.

We close with the following question that we cannot answer.

**Question.** If $G$ and $H$ are mutually embeddable non-isomorphic graphs, then do $G$ and $H$ belong to an infinite family of mutually embeddable non-isomorphic graphs?

**REFERENCES**


