Preface

You are reading a book about a game. More specifically, the game Cops and Robbers, which is played on a graph. Cops and Robbers, in the form we study it, was first introduced in the early 1980s, and a robust body of work on the topic has been growing steadily ever since. At its core, it is a game played with a set of cops (controlled by one player) trying to capture the robber (controlled by the opposing player). The cops and the robber are restricted to vertices, and they move each round to neighboring vertices. The smallest number of cops needed to capture the robber is the *cop number*. Such a simple-sounding game leads to quite a complex theory, as you will learn. A formal introduction to the game and the cop number is given in Chapter 1. Despite the fact that the game is nearly three decades old, the last five years however, have seen an explosive growth in research in the field. Some newer work settles some old problems, while novel approaches, both probabilistic, structural, and algorithmic, have emerged on this classic game on graphs.

We present a book which surveys all of the major developments (both historical and recent) on the topic of Cops and Robbers. As the moniker "Cops and Robbers" represents a class of games with varying rules, we emphasize that we primarily study the game where the cops and robber have perfect information, may only move to neighboring vertices, and move at unit speed (a player can only move at distance at most one at any step of the game). There is a large and growing literature on variants of the Cops and Robbers game, where there is some notion of "good guys" versus "bad guys". For example, there are versions where there is imperfect information, players can occupy edges or only a subset of vertices, move at faster speeds, or the cops are trying to stop or contain a fire, disease, or contaminant spreading in a graph. Although these games are not our main focus, we do discuss some variants in Chapters 8 and 9.

There are a number of reasons why we wrote this book. One of our goals was to bring together all the most important results, problems, and conjectures in one place to serve as a reference. Hence, this book will be both invaluable to researchers in the field and their students, and a one-stop shop for the major results in the field. We also wanted the book to be self-contained and readable to an advanced undergraduate or beginner graduate student; on the other hand, there are enough advanced topics to either intrigue the seasoned mathematician or theoretical computer scientist. The book is designed to be used either in a course, or for independent reading and study. The only prerequisites would be a first course in graph theory, though some mathematical maturity and some background on sets, probability, and algorithms would be helpful. One of our principal goals is to showcase the beauty of the topic, with the ultimate aim of preserving it for the next generations of graph theorists and computer scientists. We also showcase the most challenging open problems in the field. For example, Meyniel's conjecture on upper bounds for the cop number (see Chapter 3) is a deep problem which deserves to be better known.

We now give a summary of the chapters. Chapter 1 supplies all the requisite motivation, notation, basic results, and examples for what comes later. We give a lower bound of Aigner and Fromme on the cop number in terms of girth and minimum degree, and we give the asymptotic upper bound on the cop number supplied by Frankl. Along the way, we discuss guarding isometric paths, and retracts and their critical connections to the game. In Chapter 2 we consider some new and old characterizations of k-cop-win graphs. We describe in detail the classic characterization of Nowakowski and Winkler and Quilliot of finite cop-win graphs. This beautiful characterization reduces the problem to the existence of a certain ordering of the vertices, a so-called cop-win ordering. It also leads to a strategy for catching the robber called the cop-win strategy. We survey the recent characterization of graphs with cop number $k > 1$ by Clarke and MacGillivray. This characterization uses, among other things, properties of graph products. Chapter 3 is all about Meyniel's conjecture, which concerns an upper bound on the cop number in connected graphs. We give some recent upper bounds and discuss the state-of-the-art on the conjecture. We present a recent proof of the conjecture in the special case of graphs of diameter at most 2. Chapter 4 focuses on the game in graph classes and graph products. We consider bounds for the cop number and related parameters for various products, such as the Cartesian, strong, categorical, and lexicographic products. A proof of the fact that the cop number of planar graphs is at most three is given, and graphs with higher genus are also discussed.

In Chapter 5, we consider algorithmic results on computing the cop number. After an introduction to the rudiments of complexity theory and graph algorithms, we prove that the problem of computing whether the cop number is at most k is in polynomial time, if k is fixed. If k is not fixed, we sketch the proof of the recent result that the problem is **NP**-hard. In Chapter 6 we investigate the cop number in random graphs. We present results for the cop number of the binomial random graph $G(n, p)$, when p is constant, and also consider recent work on case $p = p(n)$ is a function of n. We culminate with the beautiful Zig-Zag theorem of Łuczak and Prałat, which reveals a surprising, literal twist to the behaviour of the cop number in random graphs. We finish with a study of the cop number in models for the web graph and other complex networks. In Chapter 7 we study the game of Cops and Robbers played in infinite graphs. Infinite graphs often exhibit unusual properties not seen in the finite case; the cop number in the infinite case is no exception to this. We introduce the cop density of a countable graph and show that the cop density of the infinite random graph can be any real number in $[0, 1]$. We survey the results of Hahn, Sauer, and Woodrow on infinite chordal cop-win graphs. We finish the chapter with a discussion of paradoxically large families of infinite vertex-transitive cop-win graphs. Chapters 8 and 9 consider variants of the game, and are more like surveys when compared with previous chapters. In Chapter 8, we consider the effect of changing the rules of Cops and Robbers. In particular, we consider imperfect information where the robber is partially invisible, and the inclusion of traps, alarms, and photo radar. We consider tandem-cops where cops must always be sufficiently close to each other during the course of the game. In addition, we consider a version of Cops and Robbers where the cops can capture the robber from some prescribed distance (akin to shooting the robber), and we investigate the length of time it takes for the cops to win assuming optimal play. At the heart of all the games we consider, there is the notion of a set of *good guys* trying to stop, contain, or capture a *bad guy*. Chapter 9 deals with several of these kinds of games, including firefighting, edge searching, Helicopter Cops and Robbers, graph cleaning, and robot vacuum. We conclude with a brief section on combinatorial games.

We think the book would make a solid second or topics course in graph theory. An ambitious (likely two-term) course would cover all nine chapters. For a one-term course, we suggest three options: generalist and specialist courses, along with an experiential option. A *generalist* course would cover each of the first four chapters and two of the remaining ones. Such a course would give a solid grounding in the field, and supply some flexibility at the end, depending on the tastes of the instructor and audience. A *specialist* course would cover the first two chapters, one of Chapters 3 or 4, and three of the last five chapters. This option would appeal to those would like to learn one of the more advanced topics (such as algorithms, random or infinite graphs) in greater detail. Finally, an *experiential* course would cover Chapters 1, 2, 4, 8, and 9. The emphasis in such a course would be on projects, filling in omitted proofs (Chapters 4, 8, and 9 contain surveys with proofs omitted), coming up with new examples, and developing new variants of Cops and Robbers. This last option would be especially useful in the setting of a summer research project (such as one sponsored by an NSERC USRA or NSF REU).

To both aid and challenge the reader, there are over 200 exercises in the book, with many worked examples throughout. Open problems are cited in the exercises and elsewhere. We will maintain a website

`http://www.math.ryerson.ca/~abonato/copsandrobbers.html`

which will contain resources such as errata and lists of open problems. Hopefully, it will also contain their eventual solutions!

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