Graph Protection

GRASCan Workshop, Ryerson University, May 25-27, 2012

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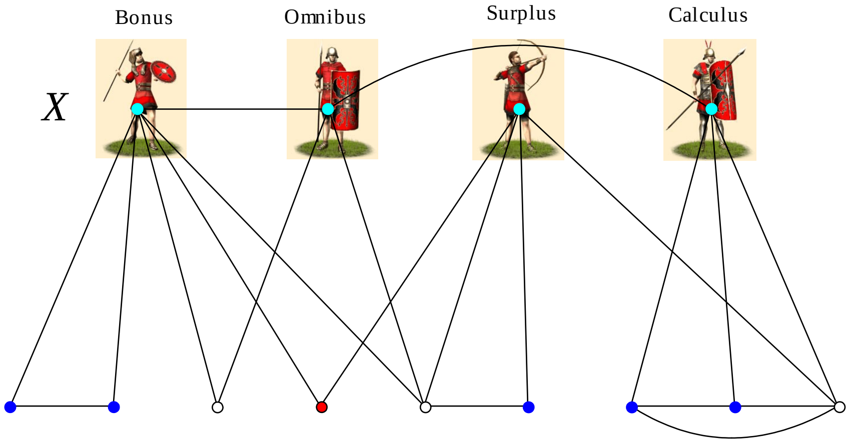
protectors	$=$	**guards (emergency vehicles, etc.)**
event	$=$	**attack (fire, illness)**

guards move along edge in **unit time interval**

Example



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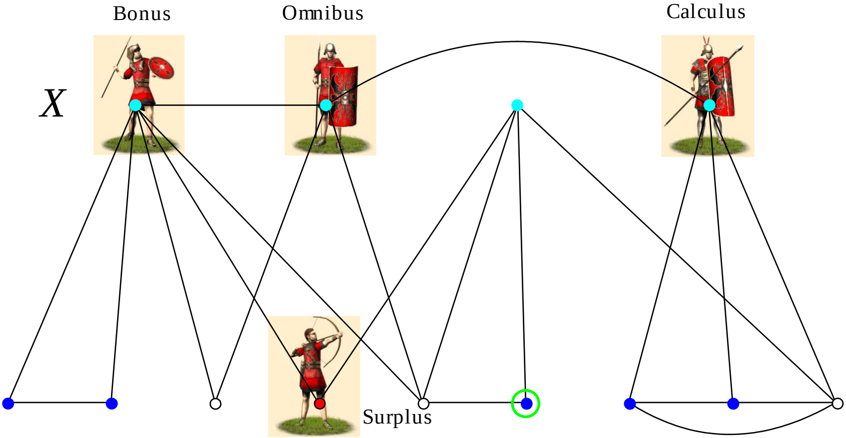

Who started it?

Roman Emperor **Constantine The Great**, 274 – 337 CE

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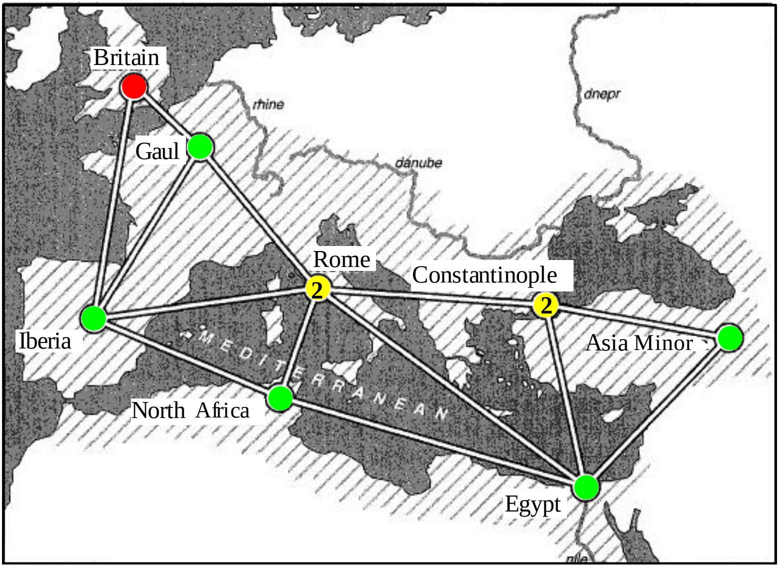

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- This strategy is called **Roman domination.**

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- **Roman domination number $\gamma_{\mathrm{R}}(G)$ of G:**
  minimum weight of a Roman dominating function.

Weak Roman Domination

Improved strategy:

- Place 0, 1 or 2 guards per vertex such that
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- This protection model is called **weak Roman domination.**

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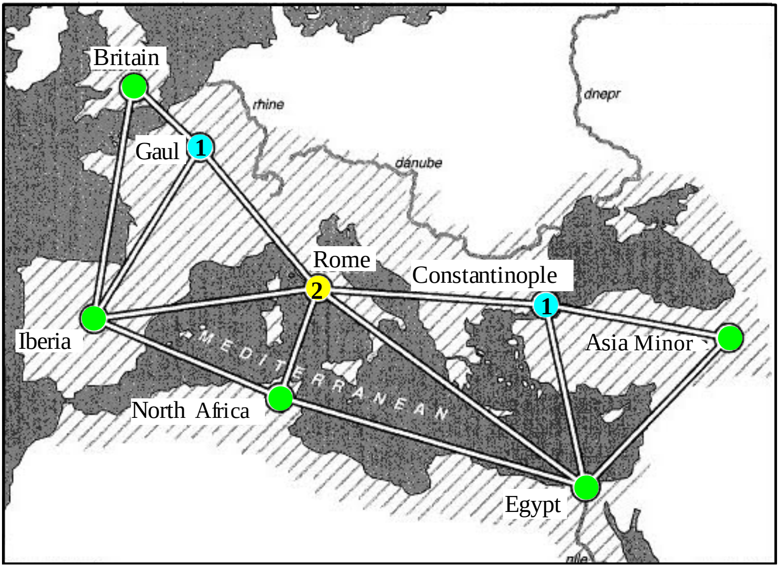

Secure Domination

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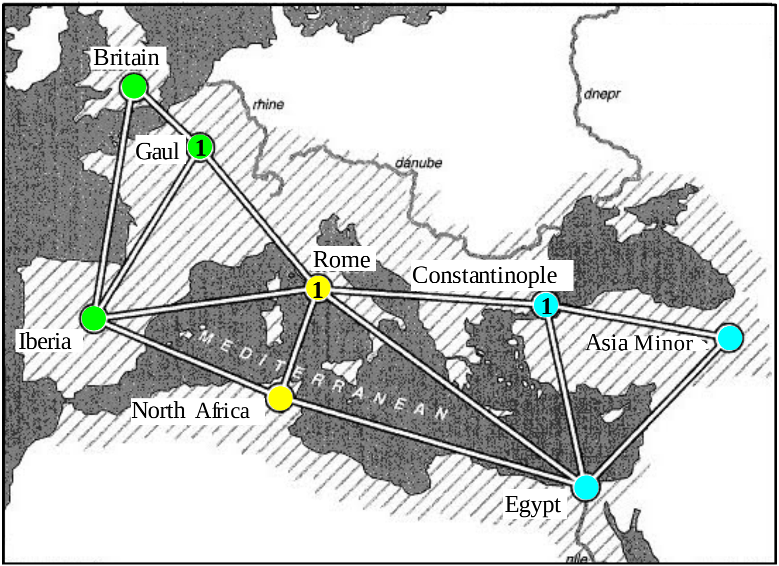
Secure Domination



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- **Secure domination number** $\gamma_s(G)$ of G:
  minimum cardinality of an SDS.

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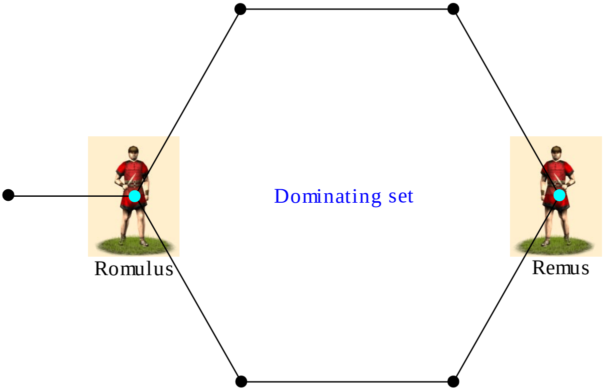

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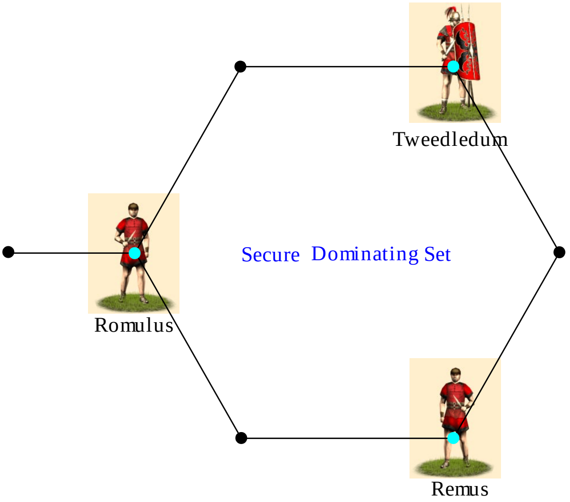

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Let $X \subseteq V$ and $u \in X$.

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- **External X-private neighbourhood** $\text{epn}(u, X)$ of u:
  set of all external X-private neighbours of u.

Secure Domination

Theorem

Let X be a dominating set of G. Vertex $u \in X$ defends $v \in V - X$ if and only if v is adjacent to each vertex of $\{u\} \cup \operatorname{epn}(u, X) - \{v\}$.

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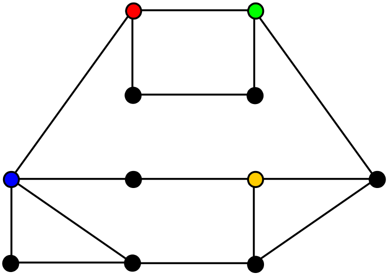

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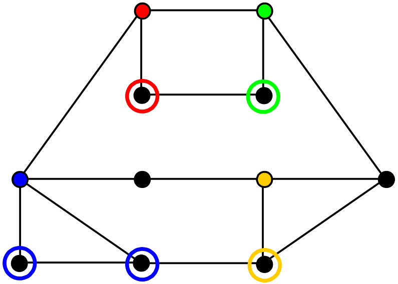

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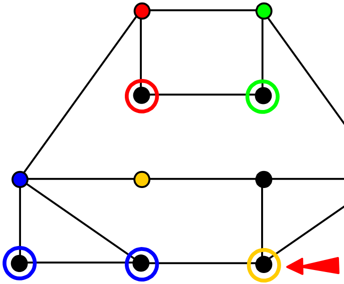

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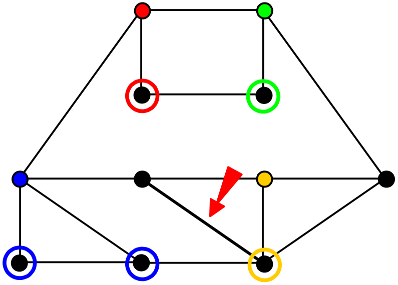

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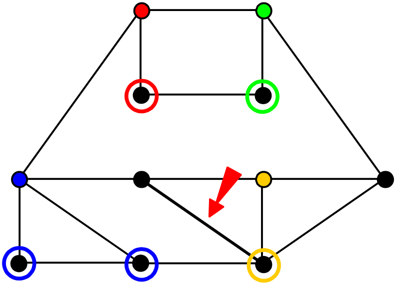

- As for Roman domination, $\gamma(G) \leq \gamma_s(G) \leq 2\gamma(G)$ for any graph G.

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4. *Is it true that* $\gamma_s(T) > \frac{1}{2}\alpha(T)$ *for all trees?*

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- $\{D_i\}$ is a sequence of vertex sets; one guard on each vertex of D_i; $\{r_i\}$ is a sequence of vertices.

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  - attacker chooses the locations of the attacks $r_1, r_2, \ldots$.
  - Defender wins if they can successfully defend any series of attacks, subject to the constraints of the game; attacker wins otherwise.

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- **Eternal domination number $\gamma^{\infty}(G)$ of G:**
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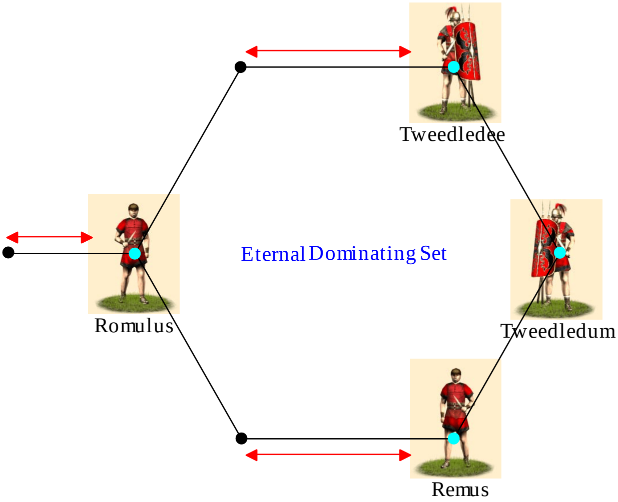

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- **Clique partition number** $\theta(G)$**:**

  smallest number of sets in clique partition,
  i.e. $\theta(G) = \chi(\overline{G})$.

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For any graph G, $\gamma^\infty(G) \leq \binom{\alpha(G)+1}{2}$ and this bound, though huge, is sharp!

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- **m-Eternal domination number $\gamma_{\mathrm{m}}^{\infty}(G)$ of G:**
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Describe classes of graphs where equality holds in each case.

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- **independent protection number** $i^{\infty}(G)$:
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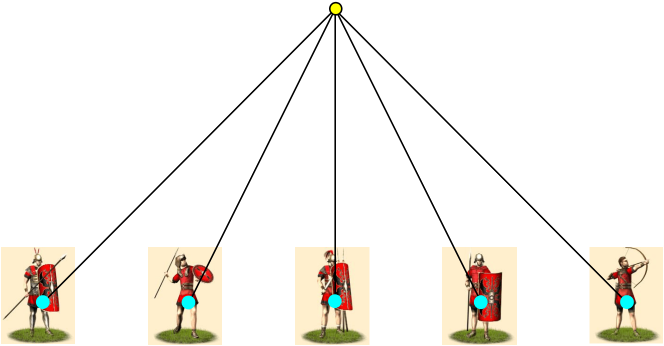
i-Protectable Graphs



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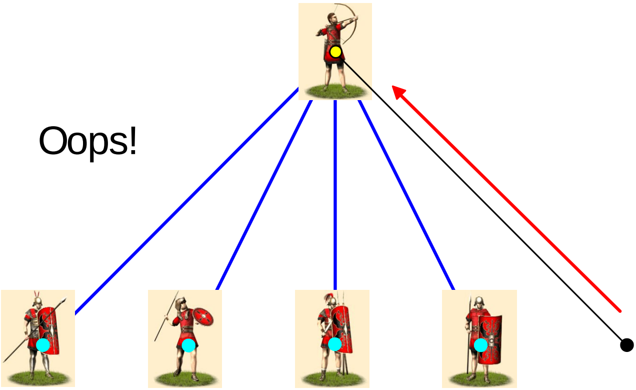

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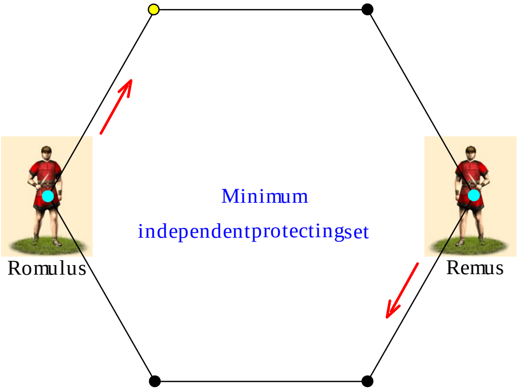

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Follows from Hall's theorem.

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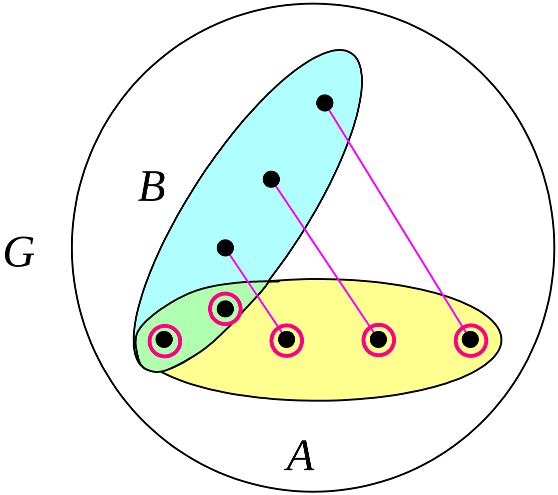

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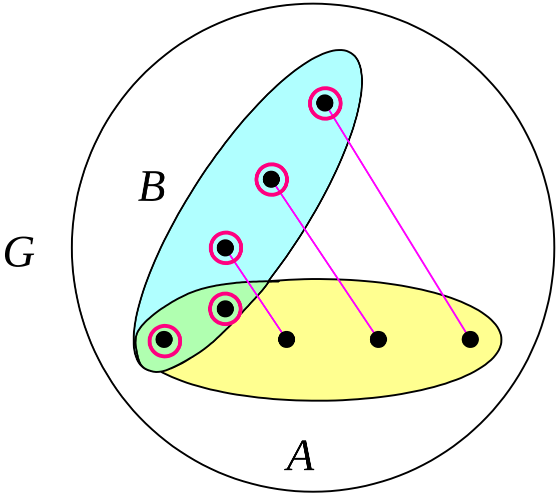

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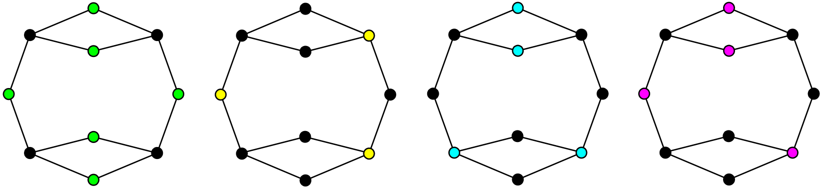

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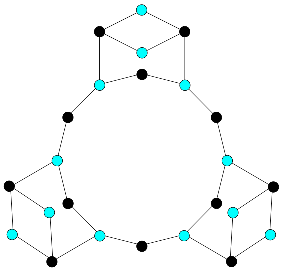
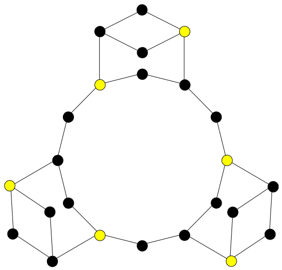
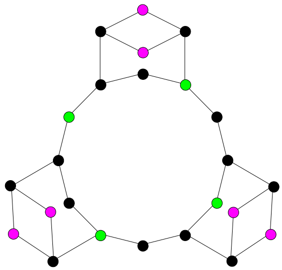

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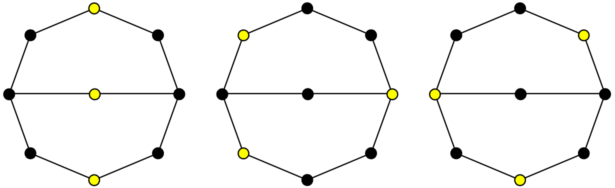

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6. *Characterize i-protectable graphs.*
   They form a nice extension of well-covered graphs.