The Petersen graph is the smallest 3-cop-win graph

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joint work with
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1: Institute for Mathematics and its Applications
2: IMA Summer Undergraduate Research Program in Interdisciplinary Mathematics (2010)

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Outline

Outline

The Game of Cops and Robbers

Introduced independently by Quilliot (1978), and Nowakowski & Winkler (1983).

How to Play Cops and Robbers

Set Up:

- Chose a graph G and a positive integer k.
- Cops $C_1, C_2, \ldots, C_k$ are placed on vertices of G
- Next, the robber R is placed on a vertex of G.

A Game Turn:

- Each cop moves to an adjacent vertex, or remains in place
- Next, the robber moves similarly.

Victory Conditions:

- Cops: one cop becomes co-located with the robber
- Robber: evades capture forever

How to Study Cops and Robbers

- Pick a graph G and a number of cops k.
- Who has a winning strategy: the k cops or the robber?

Definition

The **cop number** $c(G) =$ *fewest number of cops with a winning strategy on G.*

The cop number is always defined, since $|V|$ cops trivially win.

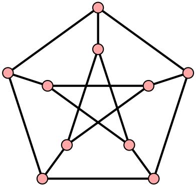
Objective: Given a graph G, determine its cop number $c(G)$.

The Petersen Graph

Lemma (Aigner and Fromme (1984))

If G has girth $g(G) \geq 5$ then $c(G) \geq \delta(G)$, the minimum degree of G.

WELL KNOWN FACT: the Petersen graph (3-regular, girth 5) is 3-cop-win.

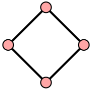
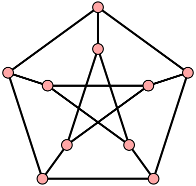


- The smallest snark
- The smallest connected, vertex-transitive graph that is not a Cayley graph
- The smallest hypo-hamiltonian graph
- The unique (3,5)-cage*

* More on this later...

Extremal Questions for Cop Number

What is the connected graph $G = (V, E)$ of smallest order with:

c(G)=1		|V|=1	
c(G)=2		|V|=4	4-cycle
c(G)=3		|V| = 10	Petersen Graph W. Baird and A. Bonato (2012+) computational proof

Baird and Bonato (2012+): Proof by Computer Search

$\|V\|$	$\# \{c(G) = 1\}$	$\# \{c(G) = 2\}$	$\# \{c(G) = 3\}$
⋮	⋮	⋮	⋮
8	3,791	7,326	0
9	65,561	195,519	0
10	2,258,213	9,458,257	1

- Determine $c(G) = 1$ by finding a cop-win ordering
- Determine $c(G) = 2$ or $c(G) = 3$ using a recursive algorithm from Bonato, Chiniforooshan and Prałat (2010).

Outline

Our Main Results [BCMMV (2012+)]

We give *mathematical proofs* of the following theorems

Theorem

If $G = (V, E)$ *is a connected graph with* $|V| \leq 9$ *then* $c(G) \leq 2$.

Theorem

Let $G = (V, E)$ *be a connected graph with* $|V| = 10$. *Then* $c(G) = 3$ *if and only if* G *is the Petersen graph. Otherwise,* $c(G) \leq 2$.

The Collaboration

UNIVERSITY OF MINNESOTA
IMA Institute for Mathematics and its Applications

Summer 2010
Interdisciplinary
Math REU

AB, Sylvia Valeva, John MacCauley, Aaron Maurer

Pursuit-Evasion Group

Paolo Codenotti

IMA Post Doctoral Fellow, 2011-2013

Our Strategy: Galactic Vertices

OUR STRATEGY: Consider $c(G)$ for graphs on n vertices with maximum degree

$$\Delta(G) \geq n - 7.$$

Recall: UNIVERSAL VERTEX

- A *universal vertex* has degree $n - 1$.
- If G has a universal vertex then $c(G) = 1$.

For this talk, we define a GALACTIC VERTEX

- A *galactic vertex* has degree at least $n - 7$.

Disclaimer: The term "galactic vertex" is for amusement purposes only, in the context of this talk. It is easier to say than "vertex with co-degree at most 7."

Notation

ASSUME: $G = (V, E)$ connected, has n vertices

Let $u, v \in V$ and $S \subset V$.

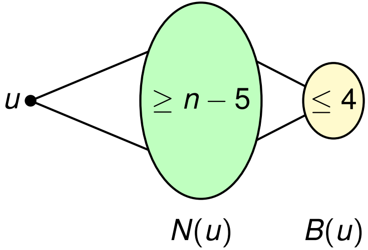
neighbors	$u \sim v$ when $(u, v) \in E$
neighborhood	$N(u) = \{v \in V : u \sim v\}$
closed neighborhood	$\overline{N}(u) = N(u) + u$
beyond set	$B(u) = V \setminus \overline{N}(u)$
degree	$\deg(v) =$ degree of v
maximum degree	$\Delta(G) = \max_{v \in V} \deg(v)$
induced subgraph	$G[S] = (S, E[S])$, where $S \subset V$

Outline

Galactic Vertex Lemma I

Lemma

If $\Delta(G) \geq n - 5$ then $c(G) \leq 2$.



forces $c(G) = 2$

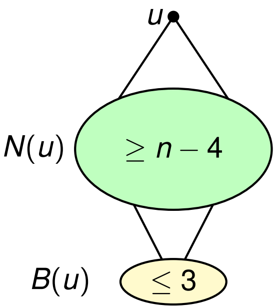
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

$\overline{N}(u) =$ closed neighborhood of u

$B(u) = V - \overline{N}(u)$

CASE: $\deg(u) \geq n - 4$

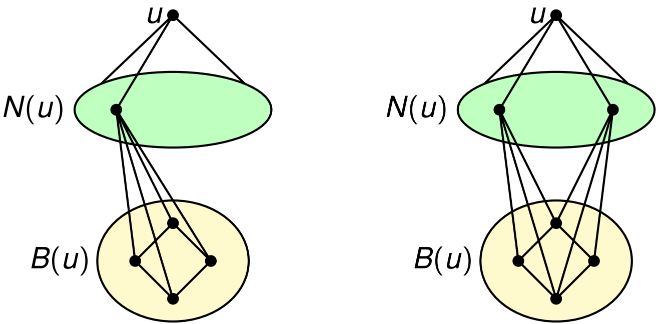
- $|B(u)| \leq n - 1 - (n - 4) = 3$
- $G[B(u)]$ is cop-win
- Winning Strategy:
  - Place C_1 on u and C_2 in $B(u)$.
  - Do not move C_1
  - Play cop-win strategy on $G[B(u)]$ with C_2



Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

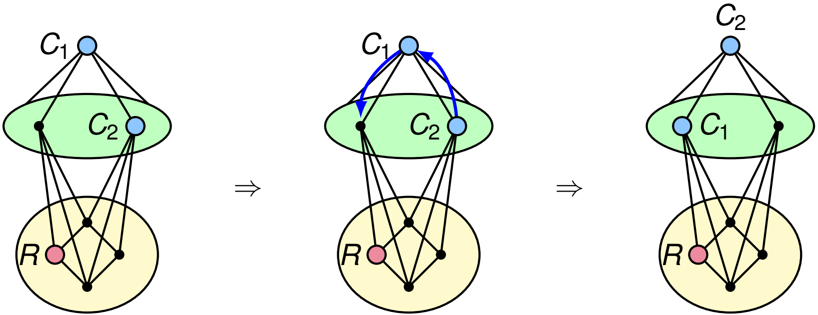
CASE: $\deg(u) = n - 5$

- $|B(u)| = n - 1 - (n - 5) = 4$
- $G[B(u)]$ must be 4-cycle (otherwise cop-win)
- Two forbidden subgraphs:



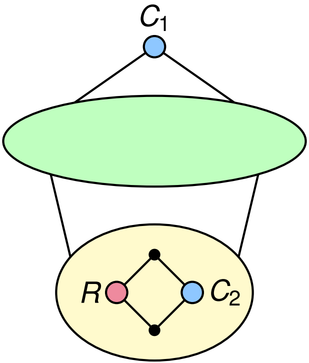
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

2-cop-win strategy for second case



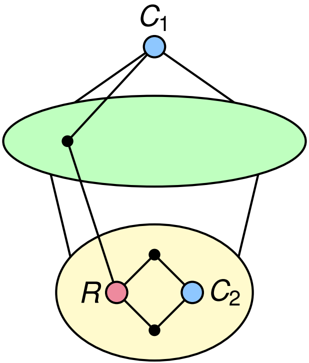
Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

The 2-cop-win strategy on G



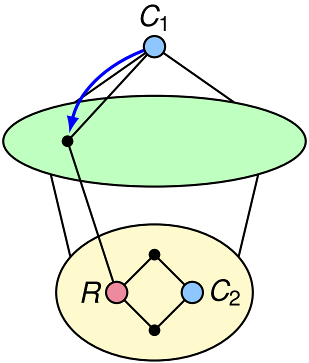
Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

The 2-cop-win strategy on G



Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

The 2-cop-win strategy on G



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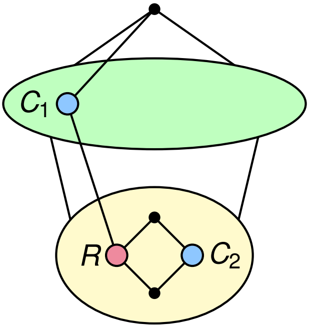
The 2-cop-win strategy on G



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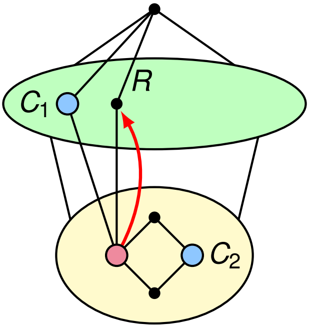
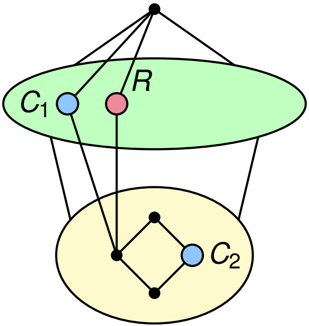
The 2-cop-win strategy on G



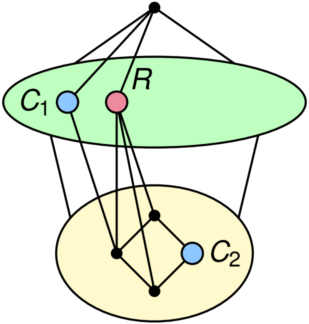
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

The 2-cop-win strategy on G



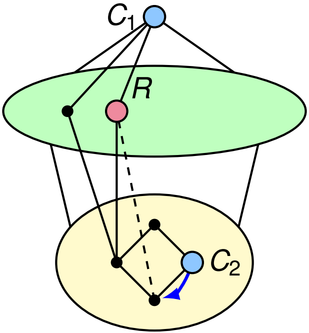
Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

Must be 3 edges from robber R to set $B(u)$



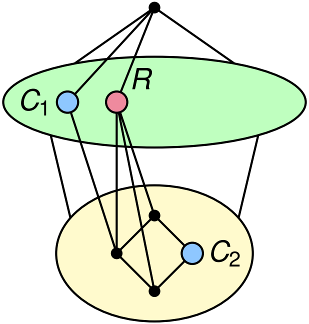
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

Otherwise: two-cop-win



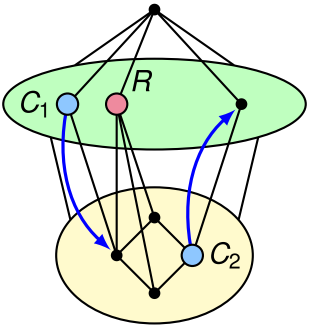
Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

Must be 3 edges from R to $B(u)$



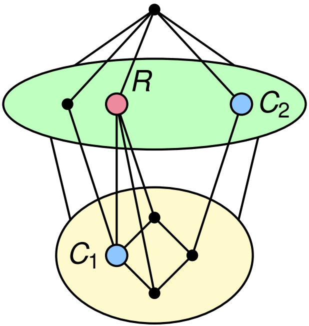
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

Cops move to threaten robber in $N(u)$



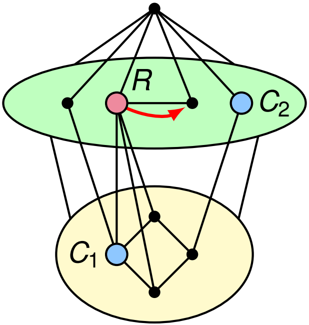
Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

Cops threaten robber in $N(u)$



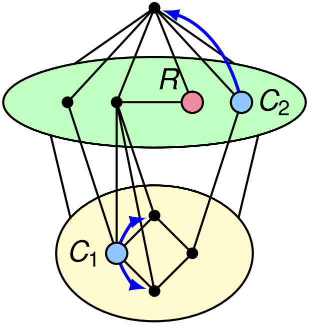
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

Robber must move within $N(u)$



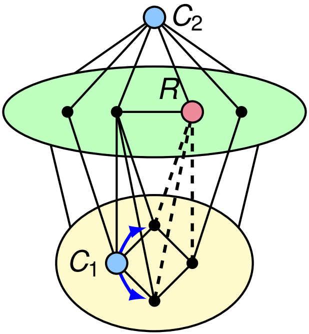
Proof : $\Delta(G) \geq n - 5 \Longrightarrow c(G) \leq 2$

Cops trap the Robber in their next move



Proof : $\Delta(G) \geq n - 5 \implies c(G) \leq 2$

$|N(R) \cap B(u)| \leq 2$, so C_1 can move to cover $N(R) \cap B(u)$. □



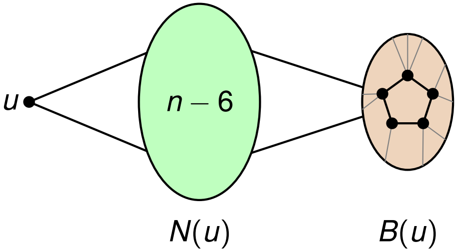
Outline

Galactic Vertex Lemma II

Lemma

Suppose that $c(G) = 3$. If $u \in V$ with $\deg(u) = n - 6$ then the induced subgraph $G[V - \overline{N}(u)]$ is a 5-cycle.

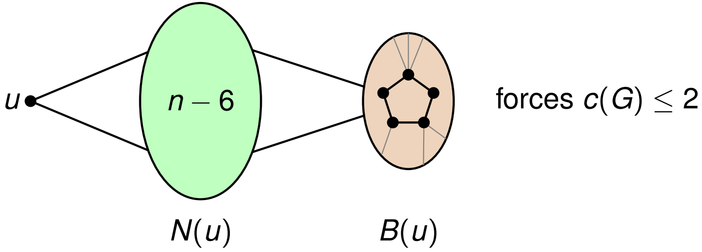
if $c(G) = 3$ then



Corollary for $\Delta(G) = n - 6$

Corollary

Suppose that $\deg(u) = n - 6$ *and* $\min_{v \in B(u)} \deg(v) \leq 3$. *Then* $c(G) \leq 2$.



Progress on Our Main Result

Galactic Lemmas I and II narrow the hunt for smallest 3-cop-win graph.

We can now prove that:

- All graphs on 9 or fewer vertices need at most 2 cops
- 10-vertex graphs with $\Delta(G) \geq 4$ need at most 2 cops

If $|V| \leq 9$ then $c(G) \leq 2$

Theorem

If $n \leq 9$ then $c(G) \leq 2$.

PROOF

- If $\Delta(G) \geq 4 = 9 - 5$, use Galactic Lemma I
  - If $n = 9$ and $\Delta(G) \geq 4$ then $c(G) \leq 2$
- If $\Delta(G) = 3 = 9 - 6$, use the Corollary to Galactic Lemma II
  - If $n = 9$ and $\deg(u) = 3$ then every $v \in V - \overline{N}(u)$ satisfies $\deg(v) \leq 3$. So $c(G) \leq 2$.
- If $\Delta(G) = 2$, then G is a path or a cycle. □

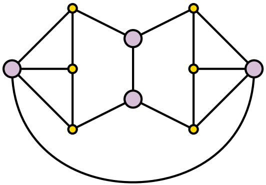
$|V| = 10$ and $\Delta(G) \geq 4$

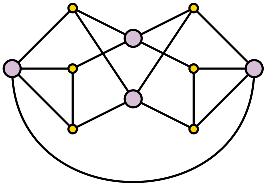
Lemma

If $n = 10$ and $\Delta(G) \geq 4$ then $c(G) \leq 2$.

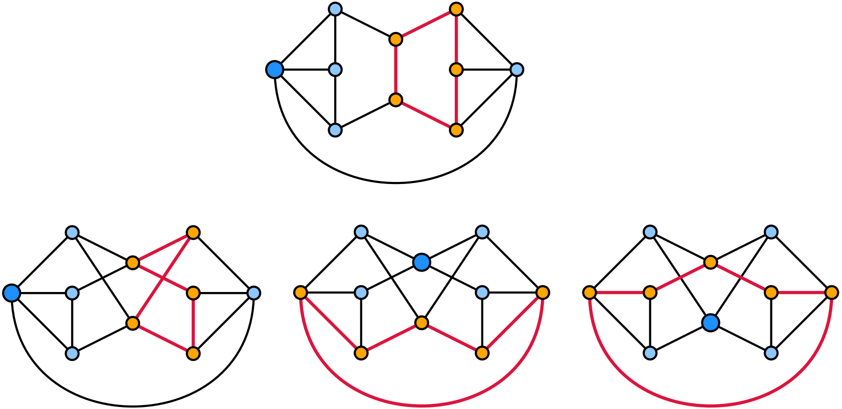
PROOF

- If $\Delta(G) \geq 5 = 10 - 5$, use Galactic Lemma I
- If $\Delta(G) = 4 = 10 - 6$, then G must contain one of the following subgraphs. This forces $c(G) \leq 2$. □





Remark: $B(u)$ is 5-cycle when $\deg(u) = 4$



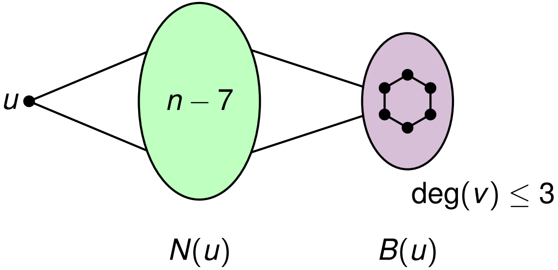
Outline

Galactic Vertex Lemma III

Lemma

Suppose that $c(G) = 3$. *If* $u \in V$ *with* $\deg(u) = n - 7$ *with* $\deg(v) \leq 3$ *for all* $v \in B(u)$. *Then the induced subgraph* $G[B(u)]$ *is a 6-cycle.*

if $c(G) = 3$ then



The Unique Smallest 3-cop-win Graph

Theorem

The Petersen graph is the only 10-vertex graph that is 3-cop win. All other 10-vertex graphs satisfy $c(G) \leq 2$.

PROOF. Suppose that $n = 10$ and $c(G) = 3$.

- Know $\Delta(G) = 3$.
- If $\deg(u) = 3$ then $G[B(u)]$ is a 6-cycle.
- G is 3-regular $\iff$ G is the Petersen graph.
- If $v \in B(u)$ has $\deg(v) = 2$, then there is a winning 2-cop strategy for G. □

Outline

The Smallest k-cop-win Graph for $k > 3$

- The Petersen graph is the smallest 3-cop-win graph.
- QUESTION: What is the smallest k-cop-win graph?

Aigner and Fromme (1984): If G has girth $g(G) \geq 5$ then $c(G) \geq \delta(G)$.

- A $(k, 5)$-cage is a k-regular graph with girth 5 of minimal order.
- The cop number of a $(k, 5)$-cage is at least k.

Cages for Robbers?

Let $k \geq 3$.

- QUESTION 1: Is a $(k, 5)$-cage k-cop-win?
- QUESTION 2: Is the $(k, 5)$-cage the smallest k-cop win graphs for $k \geq 3$?

Baird and Bonato (2012+)

- The size of the smallest k-cop-win graph is $O(k^2)$ (by construction using incidence graphs of projective planes).
- Meyniel's conjecture $\implies$ smallest k-cop win graph has order $\Omega(k^2)$

Observation

- The $(k, 5)$-cage has order $\Theta(k^2)$.
- So QUESTION 2 remains in the realm of possibility.

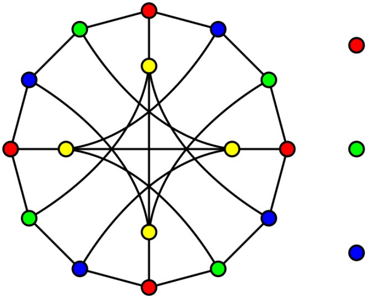
Cages for Robbers?

Thanks, Doc!

Cages

cage	n	description	order of Aut(G)
$(3,5)$	10	Petersen Graph	120
$(4,5)$	19	Robertson Graph	24
$(5,5)$	30	4 different ones	20, 30, 96 and 120
$(6,5)$	40	unique	480
$(7,5)$	50	Hoffman-Singleton Graph	252,000

The Robertson Graph



The three vertices on the right are adjacent to the vertices of the corresponding color

This unique $(4,5)$-cage is 4-cop-win.

- Start one cop on each of the three right vertices, and one on a yellow vertex.
- Move the yellow cop for the win.