PRICING TWO DIMENSIONAL DERIVATIVES UNDER
STOCHASTIC CORRELATION

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Abstract. In this paper we provide a closed-form approximation as well as a measure of the error for the price of several two-dimenional derivatives under the assumptions of stochastic correlation and constant volatility. The method is applied to the pricing of Spread Options and Quantos Options, while three models for the stochastic correlation are considered.

1. Introduction

Most models used in the pricing of multidimensional derivatives consider constant correlation among their constituents. Nevertheless, empirical facts suggest that correlation varies over time ([10]). Neglecting changes in the correlation may introduce significant misleading in the pricing.

The objective of this paper is to provide a simulations-free approximation to the price of Spread Options and Quantos Options under non-constant correlation. In particular we study three types of models for the correlation: a deterministic time function, a mean-reverting differential stochastic process and a switching Markovian model. We obtain an expression for the price in terms of the average correlation during its lifetime which allows us to use some suitable approximations, expanding the technique in [3].

Some works introducing stochastic correlation in the pricing of financial derivatives are [9] and [4]. In [9] the author provides analytical properties of the correlation process as well as an approximation to the value of Quantos options under the assumption of constant volatility for the underlying. This approximation not only lacks a set of bounds for the error but also limits the applicability of the idea to other derivatives. In [4], the price of a spread option was obtained under stochastic correlation using numerical methods based on multidimensional trees. In this paper we provide a framework for pricing two-dimensional derivatives under time dependent correlation together with a bound for the error and without the need for time-consuming numerical methods.
Examples of two-dimensional continuous time models that introduce constant correlation on leading Brownian motions have been successfully used for modeling and pricing Spread Options, see [2]. As we will see in section 3 these models have the nice property that the law of the bi-dimensional random variable representing the value of the assets at maturity time has a very tractable form. This implies that the pricing problem for European type options can be solved with relative ease, providing accurate closed-form approximations (see [3]) and even in some cases it is possible to provide exact closed forms expression for option prices, [7] and [13].

In the models considered in this paper, the correlation and volatility are treated separately and the later is assumed constant. The process studied for the correlation follows along the lines of bounded stochastic processes as those presented by [18], [19], [9]. An alternative way to obtain stochastic correlation is by introducing the dependence between asset’s prices through the covariance matrix, for example the Wishart process and the Principal component process (see [5], [12], [11]). This type of models allow for closed-form solutions to some multidimensional derivatives like Correlation Options and Best-of Basket Options due to the existence of an analytical expression for the conditional characteristic function of the underlings (see [15], [5], [11]) but in general they fail to provide analytical expressions for most existing derivatives. Our propose models are neither particular cases nor generalizations of the covariance-based models therefore finding approximations for these derivatives, as well as for Spread Options and Quantos Options, is of great importance for practitioners.

The paper is structured as follows. In section 2 we introduce the general framework for the processes and the specific derivatives to be considered throughout the paper. In section 3 we first present some known results that applied under constant correlation models. Then the key properties that allow us to provide good approximations under stochastic correlation are presented. In section 4, the goodness of the approximation is empirically tested under two toy derivatives, two models and a wide region of the parametric space, while another two families of derivatives are partially described. Section 5 concludes.
2. General Framework

Consider a bivariate price process on a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}_t, \mathcal{P})\) such that the components of the \(\mathbb{R}^2\)-valued stochastic processes \(S(t) = (S_1(t), S_2(t))\) satisfies:

1. \(dS_1(t) = S_1(t) [\mu_1(t, S(t))dt + \sigma_1(t, S_t) dW_1(t)]\)
2. \(dS_2(t) = S_2(t) [\mu_2(t, S(t))dt + \sigma_2(t, S_t) dW_2(t)]\)

where

3. \(W_2(t) = \int_0^t \rho(s, S_s) dW_1(s) + \int_0^t \sqrt{1 - \rho(s, S_s)^2} d\tilde{W}_2(s)\)

and \(W_1\) and \(\tilde{W}_2\) are two independent Brownian motions.

We will also assume that the coefficients satisfy the necessary conditions for the existence and uniqueness of solution of a system of stochastic differential equations.

It is easy to verify that \(W_2\) is a Brownian motion under \(\mathcal{P}\).

Now \(W_1\) and \(W_2\) are dependent Brownian motions with the property that infinitesimal increments of them at time \(t\) are dependent Gaussian random variables with correlation coefficient \(\rho(t, S_t)\). This fact will be expressed symbolically as:

4. \(E[dW_1(t).dW_2(t)] = \rho(t, S_t)dt\)

An important and well studied case is when the coefficients in (1) and (2) are constants:

5. \(dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t) dW_i(t), \text{ for } i = 1, 2\)

and the correlation in (4) is also constant equal \(\rho\) for every \(t\) in \([0, T]\).

The equations in (5) have an explicit solution given by:

6. \(S_i(t) = S_i(0) \exp \left[ (\mu_i - \frac{\sigma_i^2}{2}) t + \sigma_i W_i(t) \right] \text{ for } i = 1, 2\)

From now on, we will refer to this model for assets as the Geometric Brownian Motion (GBM).

A very important property of this model is that under the assumption of constant correlation the vector \(l_T = (\log S_1(T), \log S_2(T))\) has a bivariate normal distribution, in fact:

7. \(l_T \sim N \left( \left( \begin{array}{c} \mu_1 - \frac{\sigma_1^2}{2} T \\ \mu_2 - \frac{\sigma_2^2}{2} T \end{array} \right), \left( \begin{array}{cc} \sigma_1^2 T & \sigma_1 \sigma_2 \rho T \\ \sigma_2^2 T & \sigma_2^2 T \end{array} \right) \right)\)

The property that the probability law of \(l_T\) is a bivariate normal is not exclusive of the GBM model. At least another well studied model called the log-Orstein-Uhlenbeck (log-OU) satisfies this property. The model (log-OU) has the mean-reversion property so it could be used
to model bivariate financial series in the context of commodities and interest rates (see [13]).

The log-OU process satisfies:

\[ dS_i(t) = -\lambda_i (\log S_i(t) - \eta_i) S_i(t)dt + \sigma_i S_i(t)dW_i(t) \quad \text{for } i = 1, 2 \]

The solution to these equations is:

\[ S_i(t) = \exp \left( \nu_i + e^{-\lambda_i T} (\log S_i(0) - \nu_i) + \sigma_i \int_0^T e^{-\lambda_i (T-s)} dW_i(s) \right) \]

where \( \nu_i = \eta_i - \frac{\sigma_i^2}{2\lambda_i} \).

For the log-OU process, under the assumption that \( E[dW_1(t).dW_2(t)] = \rho dt \), the law of \( b_T \) is also bivariate normal with mean \( m \) and covariance matrix \( \Gamma \) given by:

\[ m = \begin{pmatrix} \nu_1 + e^{-\lambda_1 T} (\log S_1(0) - \nu_1) \\ \nu_2 + e^{-\lambda_2 T} (\log S_2(0) - \nu_2) \end{pmatrix} \]

and

\[ \Gamma = \begin{pmatrix} \sigma_1^2 \left( \frac{1-e^{-2\lambda_1 T}}{2\lambda_1} \right) & \rho \sigma_1 \sigma_2 \left( \frac{1-e^{-(\lambda_1+\lambda_2)T}}{\lambda_1+\lambda_2} \right) \\ \rho \sigma_1 \sigma_2 \left( \frac{1-e^{-(\lambda_1+\lambda_2)T}}{\lambda_1+\lambda_2} \right) & \sigma_2^2 \left( \frac{1-e^{-2\lambda_2 T}}{2\lambda_2} \right) \end{pmatrix} \]

In this work we allow the correlation \( \rho(t, S_t) \) not to be constant as in previous examples but to be in general a stochastic process. In order to simplify notation we will denote the stochastic correlation at time \( t \) as \( \rho_t \). The stochastic dynamic for \( \rho_t \) that we consider in the examples are:

- Mean reverting square root process, which was already proposed in [9] is the process satisfying the SDE:

\[ d\rho_s = a(m - \rho_s)ds + c\sqrt{1 - \rho_s^2}dB_s \]

where \( B \) is a Brownian motion independent of \( W_1 \) and \( W_2 \). Some properties of this stochastic process were obtained in [9].

- The Markov Correlation Switching model. This model is none but a continuous time Markov Chain with state space \( (\rho_i)_{i \in I} \) where \( I \) is a finite set.

These models are flexible enough to represent a large variety of correlation behaviors.
Let $\mathcal{C}_T$ be the class of European type derivatives whose payoff $h$ depends on the bivariate vector $S(T)$ where $T$ is the maturity time. Our main objective will be to price derivatives on $\mathcal{C}_T$ under the assumption of stochastic correlation. In particular we analyze, in section 4.1 and 4.2 two cases of derivatives on $\mathcal{C}_T$

- Spread Options: with payoff: $h(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+$
- Quantos Options: with payoff: $h(S_1(T), S_2(T)) = (S_1(T)S_2(T) - K)^+$

There are two other derivatives which are partially analyzed in this paper:
- Correlation Options: $h(S_1(T), S_2(T)) = (S_1(T) - K_1)^+(S_2(T) - K_2)^+$
- Best of Basket Option: $h(S_1(T), S_2(T)) = (\max\{S_1(T), S_2(T)\} - K)^+$

These last two derivatives will be studied in section 4.3.

3. Pricing Derivatives under stochastic correlation models

On this section we obtain an exact expression for the price of a derivative on $\mathcal{C}_T$ under stochastic correlation. As this expression could not be evaluated exactly in most cases, we provide some possible approximated approaches.

3.1. Exact pricing under stochastic correlation models. We assume a European type derivative on $\mathcal{C}_T$ with payoff $h$ and constant interest rate $r$. Hence the price of such derivative at time $t \leq T$ is the discounted value of the expectation of the payoff function at maturity time $T$ under the risk neutral measure $Q$:

$$\Pi_t = e^{-r(T-t)}E_Q h(S_1(T), S_2(T))$$

The probability measure $Q$ is such that the discounted processes $S_1$ and $S_2$ are $Q$-martingales. It is assumed that the market is complete and therefore this measure exist and is unique. In general, the price of a derivative in $\mathcal{C}_T$ depends on the assets model parameters only through the probability law of the vector $S(T) = (S_1(T), S_2(T))$ under $Q$. The known results showed before about the normality of $l_T$ for the GBM and the log-OU process are valid under the assumption that correlation $\rho(t, S_t)$ is constant over the whole interval $[0, T]$. But the hypothesis that instantaneous correlation remains constant over time
is not consistent with empirical facts ([9], [10]). That’s why we allow for non-constant correlation.

The main cost of introducing stochastic correlation from the point of view of pricing, is that the probability law of $l_T$ could not be obtained in closed form, even for very simple models as GBM. Here we propose a methodology for pricing that overcomes this difficulty.

For the rest of this section we will be concerned with the GMB model but it is not difficult to obtain similar results for the log-OU process following the same ideas.

A first step in the analysis is to consider that $\rho_t$ is not constant over $[0, T]$ but deterministic. In this particular case we have the following result:

**Theorem 1.**

**Proposition 2.** Let $S_1$ and $S_2$ be two stochastic processes following GBM model driven by Brownian motions $W_1$ and $W_2$ respectively. Suppose that $\rho_t$ is the instantaneous correlation coefficient between $W_1$ and $W_2$ at time $t$. Then, if $\rho_t$ is a deterministic and Riemann integrable function on $[0, T]$, then the vector $l_T$ has a bivariate normal distribution of mean

$$
\left( \frac{\mu_1 - \sigma_1^2/2}{\sigma_1^2}, \frac{\mu_2 - \sigma_2^2/2}{\sigma_2^2}, \int_0^T \rho_s ds \right)
$$

and covariance matrix

$$
\begin{pmatrix}
\sigma_1^2 T & \sigma_1 \sigma_2 \int_0^T \rho_s ds \\
\sigma_1 \sigma_2 \int_0^T \rho_s ds & \sigma_2^2 T
\end{pmatrix}
$$

The proof of this theorem is straightforward from expression 6.

One of the consequences of Theorem 2 is that the probability distribution of $l_T$ in the case of $\rho_t$ deterministic is the same probability distribution than the one assuming constant correlation equal to the mean value of the correlation over the interval $[0, T]$:

$$
\rho_T^* = \frac{1}{T} \int_0^T \rho_s ds.
$$

Let us now look closely expression (13). In general $\Pi_0$, the price at time $t = 0$ of a derivative in $C_T$, will be a function of several parameters, including volatilities, initial values for the assets, etc. In the case that the processes of the underlying $S_1(T)$ and $S_2(T)$ are driven by dependent Brownian motions with constant correlation $\rho$, then $\Pi_0$ will also depend on this correlation coefficient (see [3]). Now we look
at the price as a function of the constant value \( \rho \), all other quantities fixed, namely, \( \Pi_0(\rho) \). The function \( \Pi_0(\rho) \) will play a key role through this work, so we will refer to it as the constant correlation price function. We assume that \( \Pi_0(\rho) \) is a known function. Then the way the price is obtained in the constant correlation case could be easily extended to the case of non-constant but deterministic correlation. It is an immediate consequence of Proposition 2, expressed in the following:

**Corollary 3.** Under GBM model, if \( \rho_t \) is a deterministic and Riemann integrable function on \([0, T]\), the price of a derivative in \( \mathcal{C}_T \) is given by \( \Pi_0(\rho^*_T) \) where the function \( \Pi_0(\rho) \) is the constant correlation price function for such derivative.

**Remark 4.** As the price depends of the trajectory of \( \rho_t \) through \( \rho^*_T \), the really important hypothesis of Corollary 3 is the non-random character of \( \rho^*_T \).

In the stochastic correlation framework the previous approach is not possible because \( \rho^*_T \) is no longer a real constant value but a random variable. Nevertheless Corollary 3 has not only theoretical importance but also a practical one because it could be used to obtain an expression for the option price after conditioning on \( \rho^*_T \):

\[
p = e^{-rT}E_Qh(S_1(T), S_2(T))
\]

\[
= e^{-rT}E_Q(E_Q(h(S_1(T), S_2(T))|\rho^*_T))
\]

\[
= E_Q(e^{-rT}E_Q(h(S_1(T), S_2(T))|\rho^*_T))
\]

\[
= E_Q\Pi_0(\rho^*_T)
\]

The last equality is consequence of Remark 4. This led us to the following:

**Theorem 5.** Let \( S_1 \) and \( S_2 \) be two stochastic processes following GBM model driven by Brownian motions \( W_1 \) and \( W_2 \) respectively. Suppose that \( \rho_t \) is the instantaneous correlation coefficient between \( W_1 \) and \( W_2 \) at time \( t \). If \( (\rho_s)_{0 \leq s \leq T} \) is a stochastic process such that its trajectories are integrable function on \([0, T]\) almost surely, then the price \( p \) of a derivative in \( \mathcal{C}_T \) is obtained as:

\[
p = E_Q\Pi_0(\rho^*_T)
\]

where \( \Pi_0(\rho) \) is the constant correlation price function for such derivative.

Theorem 5 is key for pricing derivatives in the context of stochastic correlation but the usefulness could be limited when the probability
law of \( \rho^*_T \) is unknown. Therefore in the next section we provide a methodology to obtain an expression for \( E_Q \Pi_0(\rho^*_T) \) as well as bounds for the approximating error under very general conditions.

**Remark 6.** Monte Carlo (MC) simulations are always available to approximate the expression \( E_Q \Pi(\rho^*) \). Approximating \( E_Q \Pi(\rho^*) \) using MC involves the generation of correlation process trajectories and the evaluation of \( \Pi_0 \) in the trajectories mean values. With a traditional MC approach it is also necessary to generate trajectories of the underlying bivariate process \( S \) to calculate payoffs. Thus our MC method would be less time consuming than the traditional MC for two main reasons: the generation of \( S \) is more time-consuming than the evaluation of \( \Pi_0 \), it also adds another source of randomness, increasing the variance of the price.

**Remark 7.** Note that, for a finite number of simulations \( n \), MC could only provide the prices together with a standard error (a confidence interval for the true price, therefore a probability of not getting the true value of \( \alpha \) where \( \alpha \) is related to \( n \)). On the other hand the method that will be described in section 3.2 next provide a precise value and a bound (an interval) for the error with probability one of containing the true value.

### 3.2. Approximated Closed-Form Pricing.

Obtaining closed-form formulas for prices is of vital importance for practitioners, specially for multidimensional derivatives, where outputs are needed on highly frequently basis and the industry standard alternative, MonteCarlo, could take long processing times to reach them.

The approach that we propose in this section is to approximate the price in 15 using Taylor’s polynomials to describe \( \Pi(\rho) \). For some important derivatives the constant correlation price function \( \Pi \) is a smooth function , so Taylor’s Polynomials of low degree could approximate \( \Pi \) accurately (see figure ... in section 4.1 and 4.2 for the relationship between prices and the correlation). We consider here linear and quadratic approximations which, for the examples on the next section, give us the prices with high precision.

Another advantage of using Taylor’s polynomial approach is that it is possible to quantify the approximation error.

**First order approximation**
Suppose that $\hat{\Pi}$ is the Taylor linear approximation of $\Pi$ around the point $r^* = E_Q(\rho_T^*)$, i.e.

$$\hat{\Pi}(\rho) = \Pi(r^*) + \Pi'(r^*)(\rho - r^*)$$

then the first order approximated price $p_1$ satisfies:

$$p_1 = E_Q \hat{\Pi}(\rho_T^*) = E_Q(\hat{\Pi}(\rho_T^*)) = \hat{\Pi}(r^*) = \Pi(r^*) = \Pi(E_Q(\rho_T^*))$$

To estimate the approximation error we use the remainder of Taylor’s formula. We know that there exists $\tilde{\rho}$

$$\Pi(\rho) = \Pi(r^*) + \Pi'(r^*)(\rho - r^*) + \frac{1}{2}\Pi''(\tilde{\rho})(\rho - r^*)^2$$

The first two terms of this sum corresponds to $\hat{\Pi}(\rho)$ so

$$\Pi(\rho) - \hat{\Pi}(\rho) = \frac{1}{2}\Pi''(\tilde{\rho})(\rho - r^*)^2$$

Evaluating in the random variable $\rho_T^*$ and taking expectations in both members with respect to $Q$ we have

$$E_Q \Pi(\rho_T^*) - E_Q \hat{\Pi}(\rho_T^*) = \frac{1}{2}E_Q \Pi''(\tilde{\rho})(\rho_T^* - r^*)^2$$

$$|p - p_1| = \left| \frac{1}{2}E_Q \Pi''(\tilde{\rho})(\rho_T^* - r^*)^2 \right|$$

$$\leq \frac{1}{2}E_Q |\Pi''(\tilde{\rho})|(\rho_T^* - r^*)^2$$

$$\leq \frac{1}{2} \sup_{-1 < \rho < 1} |\Pi''(\rho)|E_Q(\rho_T^* - r^*)^2$$

$$= \frac{1}{2} Var_Q(\rho_T^*) \sup_{-1 < \rho < 1} |\Pi''(\rho)|$$

**Second order approximation**

The second order approximation to the derivative price is obtained using the second degree Taylor’s polynomial $\tilde{\Pi}$ around $r^*$:

$$\tilde{\Pi}(\rho) = \Pi(r^*) + \Pi'(r^*)(\rho - r^*) + \frac{1}{2}\Pi''(r^*)(\rho - r^*)^2$$

In this case the second order approximated price is

$$p_2 = E_Q \tilde{\Pi}(\rho_T^*) = \Pi(E_Q(\rho_T^*)) + \frac{1}{2}\Pi''(E_Q(\rho_T^*))Var_Q(\rho_T^*)$$

Analogously, a bound for the error of $p_2$ can be obtained using Taylor’s formula in terms of $m_3(\rho_T^*)$ the third centered absolute moment.
of $\rho_T^*$, and the third derivative of $\Pi$:

$$\left|p - p_2\right| \leq \frac{1}{6} m_3(\rho_T^*) \sup_{-1 < \rho < 1} |\Pi'''(\rho)|$$

4. Closed-Form Pricing for some derivatives

In this section we will use the approximated closed formulas for the price in (16) and (18) to price four well-known derivatives: Spread Options, Quantos Option and Correlation and Best-of-Basket Options.

Firstly we will present in detail some stochastic model for the correlation process $\rho_t$.

4.1. Models for correlation. Three stochastic dynamic structures for the correlation are proposed in this section. The aim is to consider models that reproduce empirical correlation dynamics of dependent financial series. The proposed models are Markovian and stationary. Results from Section 3 indicate that we need to compute $E_Q(\rho_T^*)$ and $Var_Q(\rho_T^*)$ therefore special attention will be placed to the first two moments of the integrated correlation.

Square root stochastic differential equation model

The dynamic structure for the correlation that we consider here is a mean reverting square root process

$$dp_s = a(m - \rho_s) ds + \sqrt{1 - \rho_s^2} dB_s$$

This model is the same proposed and applied to the pricing of a Quantos option in [9]. Also the main properties of this model are studied.

Proposition 8. Under model (12), given the value of $\rho_0$, the expectation and the variance of random variable $\rho_T^*$ are given by:

$$E(\rho_T^*) = m + (\rho_0 - m) \frac{1 - e^{-aT}}{aT}$$

$$Var(\rho_T^*) = \frac{1}{T^2} \eta(T) - E^2(\rho_T^*)$$

where $\eta(t)$ is the solution of the non-homogeneous linear differential equation:
\begin{equation}
    a \eta'(t) = -a^2 \eta(t) + \left( \frac{c_1}{a} - c_1 \right) e^{-at} + \left( \frac{c_2}{2a + c^2} - c_2 \right) e^{-(2a + c^2)t} + (\rho_0 + a mt)^2 + (c^2 - c_3)t + \left( -c_3 - \frac{c_1}{a} - \frac{c_2}{2a + c^2} \right) \tag{22}
\end{equation}

with initial condition \( \eta(0) = 0 \) and coefficients \( c_1, c_2 \) and \( c_3 \) given by:

\begin{equation}
    c_1 = -\frac{ma(\rho_0 - m)}{c^2 + a}, \quad c_3 = \frac{c^2 + 2am}{2a + c^2}, \quad c_2 = \rho_0^2 - c_1 - c_3 \tag{23}
\end{equation}

Proof: see appendix.

Remark 9. The initial value problem in 22 is solvable explicitly. It’s solution is given in the appendix.

Markov Correlation Switching model

In this case we model the correlation as a continuous time Markov-Chain taking values on a finite set \( S = \{r_1, r_2, \ldots, r_l\} \). By definition the correlation is piece-wise constant and at random times jumps to another state in \( S \). The time between jumps is assumed to be an exponential random variable of parameter \( \lambda \), so the stochastic process \( N_t \) that counts the number of jumps up to time \( t \) is a Poison process. When jumps occur, the transition probabilities of moving from state \( r_i \) to state \( r_j \) is given by the matrix \( (r_{ij})_{1 \leq i, j \leq l} \). Here we assume that \( r_{ii} = 0 \) for all \( i \). Under these conditions we can calculate the expectation and variance of \( \rho_T^* \):

Proposition 10. In a Markov Switching Model, the conditional expectation and the conditional variance of random variable \( \rho_T^* \) given that \( \rho_0 = r_i \in S \), are given by:

\begin{equation}
    E(\rho_T^*|\rho_0 = r_i) = \frac{1}{T} m_i(T), \quad \text{for } i = 1, \ldots, l \tag{24}
\end{equation}

and

\begin{equation}
    Var(\rho_T^*|\rho_0 = r_i) = \frac{1}{T^2} \left( M_i(T) - m_i^2(T) \right), \quad \text{for } i = 1, \ldots, l \tag{25}
\end{equation}

where functions \( m_i(t) \) are the solution of the following system of linear differential equations:

\begin{equation}
    m_i'(t) = \sum_{j \neq i} \lambda r_{ij} m_j(t) - \lambda m_i(t) + r_i, \quad i = 1, \ldots, l
\end{equation}
subject to the initial conditions \( m_1(0) = m_2(0) = \cdots = m_l(0) = 0 \) and the functions \( M_i(t) \) solve the system

\[
M_i'(t) = \sum_{j \neq i} \lambda r_{ij} M_j(t) - \lambda M_i(t) + 2r_i m_i(t), \quad i = 1, \ldots, l
\]

with initial conditions \( M_1(0) = M_2(0) = \cdots = M_l(0) = 0, \quad i=1,\ldots,l \)

Proof: Define \( Y(t) = \int_0^t \rho_s ds \) and let \( m_i(t) = E(Y(t)|\rho_0 = r_i) \). Then

\[
m_i(t) = E(Y(h) + (Y(t) - Y(h)))|\rho_0 = r_i) = E(Y(h)|\rho_0 = r_i) + E(Y(t) - Y(h)|\rho_0 = r_i)
\]

\[
= r_i h + o(h) + E(E(Y(t) - Y(h)|\rho_0 = r_i) P(\rho_0 = r_i)
\]

\[
= r_i h + o(h) + \sum_{j=1}^l E(Y(t) - Y(h)|\rho_0 = r_i, \rho_0 = r_j) P(\rho_0 = r_j)
\]

\[
= r_i h + o(h) + \sum_{j=1}^l m_j(t - h) P(\rho_0 = r_j)
\]

\[
= r_i h + o(h) + \sum_{j \neq i} m_j(t - h)(\lambda r_{ij} h + o(h)) + m_i(t - h)(1 - \lambda h + o(h))
\]

So

\[
m_i(t) - m_i(t - h) = r_i h + \sum_{j \neq i} m_j(t - h)\lambda r_{ij} h - \lambda hm_i(t - h) + o(h)
\]

Dividing by \( h \) and taking limits when \( h \to 0 \) we obtain (25)

Let \( M_i(t) = E(Y(t)^2|\rho_0 = r_i) \)

\[
M_i(t) = E(Y(h) + (Y(t) - Y(h))^2|\rho_0 = r_i)
\]

\[
= E(Y(h)^2 + 2Y(h)(Y(t) - Y(h)) + (Y(t) - Y(h))^2|\rho_0 = r_i)
\]

Using the additive property of the expectation and applying the previous technique we obtain (26).

The result is obvious from the definition of \( m_i(t) \) and \( M_i(t) \)

**Remark 11.** Both initial value problems satisfy the conditions for the existence and unicity of it’s solution.
Example 12. Markov Switching model with $l = 2$ states, $r_1$ and $r_2$.
For this case, $r_{12} = r_{21} = 1$ so (25) reduces to:

(29) \[ m_1'(t) = -\lambda m_1(t) + \lambda m_2(t) + r_1 \]

(30) \[ m_2'(t) = \lambda m_1(t) - \lambda m_2(t) + r_2 \]

The solution to this system of ordinary linear differential equations with initial condition $m_1(0) = m_2(0)$ is straightforward and is given by:

(31) \[ m_1(t) = C_1 e^{-2\lambda t} + C_2 t, \quad m_2(t) = -C_1 e^{-2\lambda t} + C_2 t \]

where $C_1 = (r_1 - r_2)/4\lambda$ and $C_2 = (r_1 + r_2)/2$. On the other hand, the solution to the system in (26) for this case is:

\[ M_1(t) = A_1 + A_2 t + A_3 t^2 + (A_4 + A_5 t) e^{-2\lambda t} \]
\[ M_2(t) = B_1 + B_2 + B_3 t^2 + (B_4 + B_5 t) e^{-2\lambda t} \]

where $A_1 = B_1 = -2C_1^2$, $A_2 = 2r_1 C_1$, $B_2 = -2r_2 C_1$, $A_3 = B_3 = C_2^2$, $A_4 = B_4 = 2C_1^2$, $A_5 = -2C_1 C_2$ and $B_5 = 2C_1 C_2$. Having these explicit expressions for $m_i$ and $M_i$, $i = 1, 2$, the computation of the conditional expectation and variance of $\rho_T$ given that $\rho_0 = r_i$ is immediate from Proposition 10.

The previous example could be used to model dependencies between asset prices when there exist two different behaviors, so possible interpretations could be to model differently the behaviors corresponding to “normal period” and “crisis period”, which in principle should be more accurate than assuming constant correlation in the whole interval.

4.2. Spread Options Pricing. Here we will use the approximations proposed in Section 3.2 to the pricing of a Spread Option. A spread option has a payoff

\[ h(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+ \]

The difference of $(S_1(T) - S_2(T))$ is called the spread.

Under a GBM model with constant correlation there exist no exact closed form expression for the price of a spread option except when $K = 0$, see [8]. This is the main reason for which the available literature on the subject is devoted to approximated methods like numerical integration, Monte Carlo, trees and approximated closed forms ([2], [4], [15]).

Closed form expressions for derivative prices are of special interest to practitioners for the many advantages they provide. In the particular case of spread options under the GBM model with constant correlation,
there exists some approximated closed form expressions for the price, see [2].

For simplicity in notation we have emphasized in most of the paper the dependence of the constant correlation price function \( \Pi_0 \) on \( \rho \) but in fact the price depends also on other parameters:

\[
\Pi_0(\rho) = \Pi_0(S_1(0), S_2(0), \sigma_1, \sigma_2, r, T, K, \rho)
\]

In this paper we will use the approximated closed form proposed in [3] for \( \Pi_0 \). This formula approximates the real value of the spread option in the constant correlation case with a relative error of order \( 10^{-3} \) in a wide region of the parametric space. Analogous results are also available for the (log-OU) process.

Another advantage of this closed form approach is that also closed form expressions for sensitivities are available. In particular, sensitivities of first and second order with respect to \( \rho \) are of special interest because they appear in formulas (16) and (18).

**Numerical Results**

In figure 1 we show the almost linear behavior of the constant correlation price function \( \Pi(\rho) \).

Figure 2 shows that the correlation process may have large fluctuations on time, so if we consider stochastic correlations models we must be concerned about the influence of \( \rho_0 \) on the option price. Based on that figure we consider the prices of Spread Options for different values of \( \rho_0 \). We can see that there are significative differences between prices for different values of \( \rho_0 \)

<table>
<thead>
<tr>
<th>( \rho_0 )</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>2.4683</td>
</tr>
<tr>
<td>0</td>
<td>2.1695</td>
</tr>
<tr>
<td>0.4</td>
<td>1.8597</td>
</tr>
</tbody>
</table>

The greater the impact of \( \rho_0 \) on \( E(\rho_T^*) \), the greater impact on the price. For large values of \( T \), \( E(\rho_T^*) \) will approach the historical mean \( m \) independently of \( \rho_0 \), but for small values of \( T \) and \( \sigma \) the initial correlation value \( \rho_0 \) may have a non-negligible effect on \( E(\rho_T^*) \) and so on the option’s price. Through this work we assumed that \( \rho_0 \) is known, but in real financial problems we know this assumption is not true and it should be estimated. The described effect of \( \rho_0 \) on the price points out that it is very important to have accurate estimates of the instantaneous correlation \( \rho_0 \).

4.3. **Quantos Option.** Quantos Options: with payoff: \( h(S_1(T), S_2(T)) = (S_1(T)S_2(T) - K)^+ \) (see [16])
Figure 1. Constant correlation price function and it’s linear and quadratic approximations at points $\rho_0 = -0.25, \rho_0 = 0$ and $\rho_0 = 0.25$

Figure 2. Simulated Correlation trajectory: Emmerich model

\[
\begin{align*}
h &= \exp \{(r + \sigma_{12}) \cdot T\} \cdot V(0) \Phi(d_1) - \exp \{-r \cdot T\} \cdot K \Phi(d_2) \\
V(0) &= S_1(0) S_2(0) \\
d_1 &= \frac{\ln \frac{V(0)}{K} + (2rT + \frac{\sigma_1^2 + \sigma_2^2}{2}T)}{\sigma_v \sqrt{T}} \\
d_2 &= d_1 + \sigma_v \sqrt{T} \\
\sigma_v^2 &= \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}
\end{align*}
\]
### 4.4. Other Derivatives

In this section we show the closed form solution under constant correlation for other two-dimensional derivatives. The procedure performed in previous section could be applied to these settings by taking $\Pi(\rho^*) = h(\rho^*)$. 

<table>
<thead>
<tr>
<th>Volatilities</th>
<th>1st order Price</th>
<th>2nd order Price</th>
<th>n</th>
<th>Full MC Price (Conf. interval)</th>
<th>Partial MC Price (Conf. interval)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1 = 0.2$ &lt;br&gt;$\sigma_2 = 0.2$</td>
<td>2.1742</td>
<td>2.1695</td>
<td>1000</td>
<td>2.2411 (1.8930,2.5892)</td>
<td>2.1843 (2.1667,2.1734)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10000</td>
<td>2.1394 (2.0371,2.2417)</td>
<td>2.1680 (2.1625,2.1734)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>100000</td>
<td>2.1649 (2.1321,2.1978)</td>
<td>2.1695 (2.1678,2.1712)</td>
</tr>
<tr>
<td>$\sigma_1 = 0.2$ &lt;br&gt;$\sigma_2 = 0.8$</td>
<td>11.4145</td>
<td>11.4066</td>
<td>1000</td>
<td>10.3486 (9.3781,11.3191)</td>
<td>10.4030 (11.3723,11.4338)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10000</td>
<td>11.6210 (11.2930,11.9491)</td>
<td>11.4201 (11.4105,11.4296)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>100000</td>
<td>11.3525 (11.2502,11.4558)</td>
<td>11.4076 (11.4045,11.4106)</td>
</tr>
<tr>
<td>$\sigma_1 = 0.8$ &lt;br&gt;$\sigma_2 = 0.2$</td>
<td>12.7764</td>
<td>12.7702</td>
<td>1000</td>
<td>13.1644 (11.5494,14.7793)</td>
<td>12.7610 (12.7338,12.7882)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10000</td>
<td>12.2458 (11.7092,12.7824)</td>
<td>12.7725 (12.7637,12.7814)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>100000</td>
<td>12.7852 (12.6150,12.9555)</td>
<td>12.7718 (12.7690,12.7746)</td>
</tr>
<tr>
<td>$\sigma_1 = 0.8$ &lt;br&gt;$\sigma_2 = 0.8$</td>
<td>18.3570</td>
<td>18.3142</td>
<td>1000</td>
<td>17.2804 (15.3216,19.2392)</td>
<td>18.2868 (18.2031,18.3704)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10000</td>
<td>17.6713 (17.0347,18.3080)</td>
<td>18.2984 (18.2722,18.3246)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>100000</td>
<td>18.2857 (18.0817,18.4898)</td>
<td>18.3128 (18.3045,18.3210)</td>
</tr>
</tbody>
</table>
Pricing Two Dimensional Derivatives under Stochastic Correlation

- Correlation Options: \( h(S_1(T), S_2(T)) = (S_1(T) - K_1)^+ (S_2(T) - K_2)^+ \).

\[
\exp \{rT\} \cdot h = \exp \left\{ \mu_1^* + \mu_2^* + \frac{1}{2} \left( \sigma_1^{*2} + \sigma_2^{*2} + \sigma_{12}^{*} \right) \right\} \left[ 1 + \Phi_2(d_1, d_2, \rho) - \Phi_1(d_1) - \Phi_1(d_2) \right] \\
+ K_1 K_2 \left[ 1 + \Phi_2(d_1^*, d_2, \rho) - \Phi_1(d_1^*) - \Phi_1(d_2) \right] \\
- K_1 \exp \left\{ \mu_1^* + \frac{1}{2} \sigma_1^{*2} \right\} \left[ 1 + \Phi_2(d_1^*, d_2, \rho) - \Phi_1(d_1^*) - \Phi_1(d_2) \right] \\
- K_2 \exp \left\{ \mu_2^* + \frac{1}{2} \sigma_2^{*2} \right\} \left[ 1 + \Phi_2(d_1, d_2^*, \rho) - \Phi_1(d_1) - \Phi_1(d_2^*) \right]
\]

\[
d_1 = \frac{-\ln K_1 + \mu_1^* + \sigma_1^* T}{\sigma_1^*}, \quad d_1^* = \frac{-\ln K_1 + \mu_1^*}{\sigma_1^*}, \\
d_2 = \frac{-\ln K_2 + \mu_2^* + \sigma_2^* T}{\sigma_2^*}, \quad d_2^* = \frac{-\ln K_2 + \mu_2^*}{\sigma_2^*}
\]

\[
\mu_1^* = \ln S_1(0) + \left( r - \frac{1}{2} \sigma_1^{*2} \right) T, \\
\mu_2^* = \ln S_2(0) + \left( r - \frac{1}{2} \sigma_2^{*2} \right) T, \\
\sigma_1^{*2} = \sigma_1^2 T, \quad \sigma_{12}^{*} = \rho \sigma_1 \sigma_2 T
\]

Best of Basket Option: $h(S_1(T), S_2(T)) = \max\{S_1(T), S_2(T)\} - K^+$ (see [17])

\[
h = S_1(0)\Phi_2(d_{21}, d_{k1}, p_{2/1,k/1}) + S_2(0)\Phi_2(d_{12}, d_{k2}, p_{1/2,k/2}) + K\Phi_2(d_{1k}, d_{2k}, p_{1/k,2/k}) - K \exp\{-rT\}
\]

\[
d_{ij} = -\ln \frac{S_i(0)}{S_j(0)} + \frac{\sigma^2_Q}{2} T
\]

\[
\sigma^2_Q = \sigma_1^2 + \sigma_2^2 - 2\sigma_{12}
\]

\[
d_{kj} = -\ln \frac{K}{S_j(0)} + \frac{\sigma^2_T}{2}
\]

\[
p_{1/k,2/k} = \rho_{12}
\]

\[
p_{1/2,k/2} = \frac{\sigma_2 - \sigma_1 \rho}{\sqrt{\sigma_1^2 + \sigma_2^2 - \rho_{12}\sigma_1\sigma_2}}
\]

\[
p_{2/1,k/1} = \frac{\sigma_1 - \sigma_2 \rho_{12}}{\sqrt{\sigma_1^2 + \sigma_2^2 - \rho_{12}\sigma_1\sigma_2}}
\]

Most existing 2-dim derivatives are particular cases of the fourth derivatives mentioned in this paper.

5. Conclusion

In this paper we provide a closed-form approximation as well as a measure of the error for the price of several two-dimensional derivatives under stochastic correlation and constant volatility. Three different models for the SC are considered.

6. Appendix

Proof Proposition 8: we will divide the proof in three steps.

A first step in the proof is to obtain an expression for $E(\rho_t)$ which will be done using standard methods for diffusion processes:

1-From (19) we have

\[
(33) \quad \rho_t = \rho_0 + \int_0^t a(m - \rho_s)ds + \int_0^t c\sqrt{1 - \rho_s^2}dB_s
\]

Taking expectations in both sides of this equation and interchanging integral with expectation (Fubini’s theorem) we obtain:

\[
(34) \quad E(\rho_t) = \rho_0 + amt - a \int_0^t E(\rho_s)ds
\]
Denote $e(t) = E(\rho_t)$, then $e$ satisfies

$$e'(t) = am - ae(t)$$

with the initial condition $e(0) = \rho_0$. The solution to this initial value problem is

$$e(t) = m + (\rho_0 - m)e^{-at}$$

which obviously implies (20).

2-The expression for $E(\rho_t^2)$ will be obtained using the same method we already used to get $E(\rho_t)$: Apply Ito’s formula to $g(\rho_t)$ taking $g(x) = x^2$:

$$\rho_t^2 = \rho_0^2 + 2\int_0^t \rho_s d\rho_s + \frac{1}{2} \int_0^t d\langle \rho \rangle_s$$

After some transformations we get:

$$\rho_t^2 = \rho_0^2 + c^2 t + 2am \int_0^t \rho_s ds - (c^2 + 2a) \int_0^t \rho_s^2 ds + 2 \int_0^t \rho_s \sqrt{1 - \rho_s^2} dB_s$$

Taking expectations in both sides of this equation and interchanging integral with expectation we obtain

$$E(\rho_t^2) = \rho_0^2 + c^2 t + 2am \int_0^t E(\rho_s) ds - (c^2 + 2a) \int_0^t E(\rho_s^2) ds$$

Substituting the value of $E(\rho_s)$, taking $f(t) = E(\rho_t^2)$ and applying derivatives in both sides of the equation we get:

$$f'(t) = -(2a + c^2)f(t) - 2ma(\rho_0 - m)e^{-at} + (c^2 + 2am)$$

the solution to equation (40) given the initial condition $f(0) = \rho_0^2$ is

$$f(t) = c_1 e^{-at} + c_2 e^{-(2a+c^2)t} + c_3$$

with

$$c_1 = -\frac{ma(\rho_0 - m)}{c^2 + a}, \quad c_2 = \frac{c^2 + 2am}{2a + c^2}, \quad c_3 = \rho_0^2 - c_1 - c_3$$

3-Denote $Y_t = \int_0^t \rho_s ds$ and $\eta(t) = E(Y_t^2)$, then

$$\eta'(t) = 2E(Y_t \rho_t)$$

which obviously implies that $\eta(0) = \eta'(0) = 0$

3-On the other hand, from the model definition we have

$$\rho_t + aY_t = \rho_0 + amt + c \int_0^t \sqrt{1 - \rho_s^2} dB_s$$
Applying squares in both sides of the equation and then taking expectations we get:

\[(45)\]
\[E(\rho_t^2) + 2aE(Y_t \rho_t) + a^2E(Y_t^2) = (\rho_0 + amt)^2 + c^2E\left(\int_0^t (1 - \rho_s)ds\right)\]

rewriting this expression in terms of \(\eta(t)\) gives us:

\[(46)\]
\[E(\rho_t^2) + a\eta'(t) + a^2\eta(t) = (\rho_0 + amt)^2 + c^2t - \int_0^t E(\rho_s^2)ds\]

Substituting the value of \(E(\rho_t^2)\) on this equation we obtain the initial value problem in (22). The solution to an initial value problem of the form:

\[(47)\]
\[a\eta'(t) = -a^2\eta(t) + A_0 + A_1t + A_2t^2 + A_3e^{-at} + A_4e^{-(2a+c^2)t}\]

provided \(\eta(0) = \eta'(0) = 0\) is given by

\[(48)\]
\[\eta(t) = D_0 + D_1t + D_2t^2 + (D_3 + D_4t)e^{-at} + D_5e^{-(2a+c^2)t}\]

where

\[(49)\]
\[D_0 = \frac{A_0}{a^2} - \frac{A_1}{a^3} + \frac{2A_2}{a^4}, \quad D_1 = \frac{A_1}{a^2} - \frac{2A_2}{a^3}, \quad D_2 = \frac{A_2}{a^2}\]

\[(50)\]
\[D_3 = -\frac{A_0}{a^2} + \frac{A_1}{a^3} - \frac{2A_2}{a^4} + \frac{A_4}{a(a+c^2)}, \quad D_4 = \frac{A_3}{a}, \quad D_5 = -\frac{A_4}{a(a+c^2)}\]

REFERENCES

[17] Book in Alex Disk