An intensity-based approach for modeling hedge fund equity

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Abstract

This paper analyzes an intensity based approach for modeling hedge fund (HF) equity. We use the Cox-Ingersoll-Ross (CIR) process to describe the intensity of the HF’s default process. The intensity is purposely linked to the assets of the HF and consequently is also used to explain the equity. We examine two different approaches to link assets and intensity and derive closed-form expressions for the firms’ equity in both models. We use the Kalman filter to estimate the parameters of the unobservable intensity process. The applicability of the presented methods is demonstrated on real data working with historical series from Merrill Lynch.

Key Words: Hedge fund, reduced-form model, default process, Kalman filter.

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1 Introduction

There are mainly two classes of models in the literature that have their aim in describing and estimating the credit risk of a specific company. These two models are often referred to as structural and reduced-form (or intensity-based) models. In the last decade a new type of models, the so-called hybrid models appeared. They attempt to combine structural and intensity-based models and therefore gain from their advantages and try to eliminate their drawbacks at the same time.

According to the first structural model provided by Merton (1974), structural models have the economic interpretation of that default occurs when the value of the assets falls below the value of the company’s debt at the time of servicing the debt. Thereafter a lot of extensions were developed, such as by Black and Cox (1976) allowing the default at any time before debt matures or by Finkelstein (2001) where the default threshold is modeled as a random variable.

Reduced-form models do not consider the relation between default and a firm’s assets in an explicit manner. In these models, default is determined as the first jump of an exogenously given jump process. While in the structural models the credit-worthiness of the firm is linked to its economic and financial condition, i.e. default is endogenously generated, in intensity-based models defaults are generated exogenously. In this paper we concentrate on intensity-based models. A corresponding structural approach can be found in Escobar et al. (2008) or Escobar et al. (2009).

This paper is organized as follows. In Chapter 2 we present our two general models for pricing a hedge fund’s equity. In the first model we do not include the debt of the hedge fund explicitly, whereas in the second one the payoff to the shareholders depends directly on the company’s assets and debt. We use the theory of ordinary differential equations to derive a solution in the first model and the mechanism of Green’s function in the second. In Chapter 3 we concentrate on the numerical methods in our analysis especially on the Kalman Filter used to estimate the parameters of the CIR process. Thereafter, in Section 4.1, we describe the data used in this study. Section 4.2 is dedicated to the parameter estimation and the application of the presented models on real data. Finally, we summarize the most important findings in the Section 5.
2 Model setup

In this section we present two models that allow us to express the equity of a hedge fund as a derivative on its assets within a reduced-form methodology. We will start with a simplified version and then proceed to the more general case, capturing the leverage effect and the complete behavior of hedge funds.

For a mathematical formulation, let $T^*$ be an arbitrary but finite planning horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Furthermore, let $N = \{N(t)\}_{t \geq 0}$ be a counting process, i.e. a non-decreasing, integer-valued process with $N(0) = 0$, and let the default time $\tau_D$ be defined as the first jump of $N$, i.e. $\tau_D = \inf \{t > 0 : N(t) > 0\}$. We assume that the filtration $\mathbb{F}$ is generated by the counting process $N$ and satisfies the usual conditions of completeness and right-continuity. Setting $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$, we assume that the counting process $N = \{N(t)\}_{t \geq 0}$ admits a non-negative $\mathcal{F}_t$-predictable intensity $\lambda(t)$. For a detailed definition and a proof of the existence see Bremaud (1981). Following Bremaud (1981) and Duffie et al. (1996), the conditional default probability $\mathbb{P}^D(t, T|\mathcal{F}_t)$ for the maturity time $T$ at time $t$ is given by

$$\mathbb{P}^D(t, T|\mathcal{F}_t) = 1 - \mathbb{E}_\mathbb{P} \left[ e^{-\int_t^T \lambda(s) ds} \bigg| \mathcal{F}_t \right].$$

(1)

The corresponding conditional survival probability

$$\mathbb{P}^S(t, T|\mathcal{F}_t) = \mathbb{P}(N(T) = 0|\mathcal{F}_t) = \mathbb{P}(\tau_D > t|\mathcal{F}_t)$$

is given by

$$\mathbb{P}^S(t, T|\mathcal{F}_t) = \mathbb{E}_\mathbb{P} \left[ e^{-\int_t^T \lambda(s) ds} \bigg| \mathcal{F}_t \right].$$

We assume that

$$d\lambda(t) = \alpha(\beta - \lambda(t))dt + \sigma \sqrt{\lambda(t)}dW(t)$$

(2)

and that the interest rate $r$ is constant. For the pricing of financial derivatives, we furthermore assume the existence of a risk-neutral probability measure $\mathbb{Q}$. We introduce the risk premium as $\gamma(t) = \tilde{\gamma} \sigma \sqrt{\lambda(t)}$ and change the Wiener process via $d\tilde{W}(t) = dW(t) + \gamma(t)dt$ as well as the measure from $\mathbb{P}$ to $\mathbb{Q}$ using the
Radon-Nikodym derivative
\[
\left. \frac{dQ}{d\mathbb{P}} \right|_{\mathcal{F}_t} = \exp \left( - \int_0^t \gamma(s) dW(s) - \frac{1}{2} \int_0^t \gamma^2(s) ds \right).
\]

Thus, the intensity dynamics under $Q$ can be described by
\[
d\lambda(t) = (\alpha \beta - \lambda(t)(\alpha + \bar{\gamma} \sigma^2)) dt + \sigma \sqrt{\lambda(t)} d\hat{W}(t).
\]

Denoting $\eta = \bar{\gamma} \sigma$, we get
\[
d\lambda(t) = (\alpha \beta - \lambda(t)(\alpha + \eta)) dt + \sigma \sqrt{\lambda(t)} d\hat{W}(t).
\]  (3)

Hence, besides the real-world parameters $(\alpha, \beta, \sigma)$ one additional parameter $\eta$ has to be estimated. Equation (3) still represents the CIR process and can be rewritten in more natural form:
\[
d\lambda(t) = \hat{\alpha}(\hat{\beta} - \lambda(t)) dt + \sigma \sqrt{\lambda(t)} d\hat{W}(t),
\]  (4)

where $\hat{\alpha} = \alpha + \eta$ and $\hat{\beta} = \frac{\alpha \beta}{\alpha + \eta}$ are the risk-neutral parameters of the CIR process.

We treat the hedge fund’s equity as a derivative on the underlying hedge fund’s assets. Expectations under the risk-neutral measure are denoted by $\mathbb{E}$, i.e. we skip the subscript $Q$ of $\mathbb{E}_Q$. Thus, denoting the value of the assets at time $T$ by $A(T)$, we conclude (see, e.g. Duffie et. al. (2000)):
\[
S(t) = \mathbb{E}[e^{-r\tau} 1_{\tau > T} f(A(T)) | \mathcal{F}_t] = e^{-r\tau} \mathbb{E}[e^{-\int_0^T \lambda(s) ds} f(A(T)) | \mathcal{F}_t].
\]  (5)

Here and in the sequel we will use the abbreviation $\tau := T - t$. To specify the model, we link the expression $f(A(T))$ to the underlying intensity. We differentiate between two models: one with and one without leverage constraint.

### 2.1 Model without leverage constraint

For simplification purposes we first assume
\[
f(A(T)) = A(T) = a \cdot e^{-b \lambda(T)},
\]

where $a$ and $b$ are two positive constants, representing the fact that the higher the intensity (the higher default probability) the lower the value of the assets. In
2.1 Model without leverage constraint

In this model we do not explicitly take the firm’s liabilities into account. We apply the framework of Duffie et al. (2000), to derive the corresponding closed-form expressions for the necessary expectations. We rewrite the equations provided by Duffie in our notation and evaluate for the hedge fund’s equity by

\[ S(t) = a \cdot e^{\theta_0(\tau) + \theta_1(\tau) \cdot \lambda(t)} \cdot e^{-\tau r} \]  

where \( \theta_0(\tau) \) and \( \theta_1(\tau) \) satisfy the following ODEs:

\[
\begin{align*}
\dot{\theta}_1(\tau) &= -1 - \alpha \theta_1 + \frac{1}{2} \theta_1^2 \sigma^2, \\
\dot{\theta}_0(\tau) &= \alpha \beta \theta_1,
\end{align*}
\]

with boundary conditions \( \theta_1(0) = -b \) and \( \theta_0(0) = 0 \). Solving this system of differential equations we get:

\[
\begin{align*}
\theta_0(\tau) &= \alpha \beta \frac{\alpha - h}{\sigma^2} \tau - \frac{2 \alpha \beta}{\sigma^2} \ln \left( \frac{c + e^{-h \tau}}{e + 1} \right) \\
\theta_1(\tau) &= ce^{h \tau} \left( \frac{\alpha - h}{\sigma^2} + \frac{\alpha + h}{ce^{h \tau} + 1} \right),
\end{align*}
\]

where \( h = \sqrt{\alpha^2 + 2 \sigma^2} \)

and \( c = \frac{h + \sigma^2 b + \alpha}{h - \sigma^2 b - \alpha} \).

We furthermore show in Appendix 6.1 that these formulas represent the generalized case of a well-known pricing formula for zero-coupon bonds when the short-term interest rate follows a CIR process.

Using Equation (6), the hedge fund’s equity and its assets have the following reverse dependence:

\[
\lambda(t) = \frac{r \tau + \ln(S(t)/a) - \theta_0(\tau)}{\theta_1(\tau)}.
\]

Another important result can be perceived from expressions (5) and (6). We let \( a = 1, b = 0 \) and assume no discounting factor. Then, the closed-form solution for
the probability of default (compare with (1)) can be derived:

\[ \mathbb{P}^D(t, T) = 1 - e^{\tilde{\theta}_0(\tau) + \tilde{\theta}_1(\tau) \lambda(t)}, \quad (12) \]

where \( \tilde{\theta}_0(\tau) \) and \( \tilde{\theta}_1(\tau) \) can be calculated as above but with \( c \) defined as:

\[ c = \frac{h + \alpha}{h - \alpha}. \]

### 2.2 Model with leverage constraint

In this section we extend the previous model by introducing the leverage constraint \( X \). We define \( f(A(T)) \) by

\[ f(A(T)) = \max(A(T) - X, 0), \]

where

\[ A(T) = a \cdot e^{-\lambda(T)b} \]

as in the first model corresponds to the assets of the company and \( X \) to the company’s debt that is assumed to be constant. Hence, we can calculate the hedge fund’s equity by taking the following expectation

\[ S(t) = e^{-rt} \mathbb{E}[e^{-\int_t^T \lambda(s) ds} \max(ae^{-\lambda(T)b} - X, 0)|\mathcal{F}_t] \quad (13) \]

Using the Green’s function, \( G(\lambda, t, \lambda', T) \), we derive the closed-form solution for the hedge fund’s equity defined above. The explicit form of Green’s function for the CIR process, as presented in Buettler, Waldvogel (1996), is

\[ G_{CIR}(\lambda, t, \lambda', T) = P(\lambda, \tau)2p(\tau)H(2p(\tau)\lambda'|2k + 2, f(\lambda, \tau)), \quad (14) \]
2.2 Model with leverage constraint

\[ \tau = T - t, \quad (15) \]
\[ h = \frac{2\alpha\beta}{\sigma^2} - 1, \quad (16) \]
\[ p(\tau) = \frac{d_2(\tau)}{\sigma^2 d_1(\tau)}, \quad (17) \]
\[ f(\lambda, \tau) = \frac{8h^2 e^{h\tau} \lambda}{\sigma^2 d_2(\tau)d_1(\tau)}, \quad (18) \]
\[ P(\lambda, \tau) = B_1(\tau)e^{-B_2(\tau)\lambda}, \quad (19) \]
\[ B_1(\tau) = \left( \frac{2h e^{(\alpha+h)\frac{\tau}{2}}}{d_2(\tau)} \right)^2 \alpha\beta, \quad (20) \]
\[ B_2(\tau) = \frac{2d_1(\tau)}{d_2(\tau)}, \quad (21) \]
\[ h = \sqrt{\alpha^2 + 2\sigma^2}, \quad (22) \]
\[ d_1(\tau) = e^{h\tau} - 1, \quad (23) \]
\[ d_2(\tau) = (h + \alpha)e^{h\tau} + (h - \alpha), \quad (24) \]
\[ H(x|v, w) = \frac{1}{2} \left( \frac{x}{w} \right)^{(v-2)/4} e^{-(w+x)/2} I_{(v-2)/2}(\sqrt{wx}). \quad (25) \]

where \( I_y(\cdot) \) denotes the modified Bessel function of the first kind of order \( y \).

Given this definition of the Green’s function, Equation (13) can be represented via two integrals over the Green’s function and \( y(A(T)) \):

\[ S(t) = e^{-rt} \int_0^{\lambda^*} G_{CIR}(\lambda, t, \lambda', T) ae^{-\lambda b} d\lambda' \]
\[ \quad - e^{-rt} X \int_0^{\lambda^*} G_{CIR}(\lambda, t, \lambda', T) d\lambda' \]
\[ = e^{-rt} \left( I_1 - I_2 \right). \quad (26) \]

The integral limit \( \lambda^* \) is defined such that \( ae^{-\lambda b} - X = 0 \), i.e. \( \lambda^* = \frac{1}{b} \ln \left( \frac{a}{X} \right) \).
Substituting (14) in the expression for the second integral, we get

\[
I_2 = X \int_{0}^{\lambda^*} P(\lambda, \tau) 2p(\tau)H(2p(\tau)\lambda'|2k + 2, f(\lambda, \tau)) d\lambda'
\]

\[
= XP(\lambda, \tau) \int_{0}^{2p(\tau)\lambda^*} H(y|2k + 2, f(\lambda, \tau)) dy
\]

\[
= XP(\lambda, \tau) \chi^2(2p(\tau)\lambda^*|2k + 2, f(\lambda, \tau)).
\]

Next, we use the fact that \( H(x|v, w) \) is the density function of a non-central Chi-square distribution \( \chi^2(x|v, w) \). Analogously, for the first integral we have

\[
I_1 = aP(\lambda, \tau) \int_{0}^{\lambda^*} 2p(\tau)H(2p(\tau)\lambda'|2k + 2, f(\lambda, \tau)) e^{-\lambda'k} d\lambda'
\]

\[
= aP(\lambda, \tau) \int_{0}^{2p(\tau)\lambda^*} H(y|2k + 2, f(\lambda, \tau)) e^{-\frac{y^2}{2\tau}} dy
\]

\[
= aP(\lambda, \tau) \int_{0}^{2p(\tau)\lambda^*} \frac{1}{2} \left( \frac{y}{f(\lambda, \tau)} \right)^{k/2} e^{-\frac{f(\lambda, \tau)}{2} - \frac{y^2}{2\tau}} I_k(\sqrt{f(\lambda, \tau)y}) dy
\]

Setting \( m(\lambda, \tau) = f(\lambda, \tau) \cdot \frac{p(\tau)}{\tau} + b \) and \( l = y \cdot \frac{p(\tau) + b}{\tau} \), we get

\[
I_1 = aP(\lambda, \tau) \left( \frac{p(\tau)}{p(\tau) + b} \right)^{k+1} e^{-\frac{f(\lambda, \tau)}{2} - \frac{b}{\tau}} \int_{0}^{2(p(\tau) + b)\lambda^*} \frac{1}{2} \left( \frac{l}{m} \right)^{k/2} e^{-\frac{m+l}{2}} I_k(\sqrt{ml}) dl
\]

\[
= aP(\lambda, \tau) \left( \frac{p(\tau)}{p(\tau) + b} \right)^{k+1} e^{-\frac{f(\lambda, \tau)}{2} - \frac{b}{\tau}} \chi^2(2(p(\tau) + b)\lambda^*|2k + 2, m(\lambda, \tau)).
\]

Summarizing our findings, the hedge fund’s equity is given by

\[
S(t) = e^{-rt} P(\lambda, \tau) \left( a \left( \frac{p(\tau)}{b + p(\tau)} \right)^{k+1} e^{-\frac{f(\lambda, \tau)}{2} - \frac{b}{\tau}} \chi^2(2(p(\tau) + b)\lambda^*|2k + 2, m(\lambda, \tau)) \right)
\]

\[
- \chi^2(2p(\tau)\lambda^*|2k + 2, f(\lambda, \tau))
\]

with \( k = \frac{2p(\tau)}{\sigma^2} - 1 \) and \( \lambda^* = \frac{1}{b} \ln \left( \frac{\sigma^2}{\lambda} \right) \). In contrast to the previous model, we cannot easily transform the formula above to calculate the intensity from the equity value. Instead, we consider the equity value as a function of the intensity, given the required parameters, and then minimize the sum of the squared deviations between the given and the calculated equity value for the different data points. We obtain
the requested intensity as a solution of this minimization problem.

The model derived above also includes the first model (without leverage constraint). To see this, let $X = 0$ and receive $f(A(T)) = a \cdot e^{-b \lambda(T)}$. The closed-form solution of the second model then simplifies to

$$S(t) = a \cdot e^{-r \tau} P(\lambda, \tau) \left( \frac{p(\tau)}{b + p(\tau)} \right)^{k+1} e^{-\frac{r}{2} \frac{b}{p(\tau) + \lambda}}. \quad (30)$$

Here, we used the fact that $\lambda^*$ tends towards infinity when $X \to 0$ and that the probability function of the Chi-square distribution equals 1 when the argument reaches infinity. In Appendix 6.2 we show that for this case both analytical expressions for the first and the second model coincide.

## 3 Procedure

In the previous chapter we have shown how we can model the hedge fund’s equity within a reduced-form model. The remaining question is how to estimate the unknown parameters including the three coefficients from the CIR process, $\alpha, \beta$ and $\sigma$, as well as the coefficients $a$ and $b$ that we use to link the assets with the intensity. Hereby, we assume the leverage level to be known and constant.

For this estimation we use an extended Kalman filter. We first formulate the state-space problem where the equity time series is an observable process and the underlying intensity is an unobservable state process that should be estimated. We base our derivation on the state-space approach described by Geyer and Pichler (1999).

We use real-world observations of the equity process which is why we describe the intensity dynamics under the real-world measure $\mathbb{P}$. However, the pricing should be done under the risk-neutral measure $\mathbb{Q}$. Hence, besides the real-world parameters ($\alpha, \beta, \sigma$) one additional parameter $\eta$ must be estimated. To summarize, we will use the real-world measure $\mathbb{P}$ and Equation (2) in the prediction step of the Kalman filter and the risk-neutral probability measure $\mathbb{Q}$ and Equation (3) or (4) for the correction step. Since the use of the CIR process leads to a non-Gaussian model, the standard Kalman filter is no longer optimal and cannot be applied in its original form. To overcome this issue we replace the exact transition density
(non-central $\chi^2$) by a normal density:

$$\lambda_t|\lambda_{t-1} \sim N(\mu_t, Q_t).$$

We define $\mu_t$ and $Q_t$ within a moment matching method, so that the first two moments of the exact and approximated density function coincide:

$$\mu_t = \beta(1 - e^{-\alpha \Delta t}) + e^{-\alpha \Delta t} \lambda_{t-1}$$

and

$$Q_t = \sigma^2 \frac{1 - e^{-\alpha \Delta t}}{\alpha} \left[ \beta \frac{1 - e^{-\alpha \Delta t}}{2} + e^{-\alpha \Delta t} \lambda_{t-1} \right] + e^{-2\alpha \Delta t} Q_{t-1}.$$

Both equations help to construct the so-called prediction step. The value of the unobservable factor $\lambda$ at time $t$ based on its value at time $t - 1$ is given by

$$\lambda_{t|t-1} = \beta(1 - e^{-\alpha \Delta t}) + e^{-\alpha \Delta t} \lambda_{t-1|t-1},$$

with $\Delta t$ representing the time step. Analogously, we get for the variance:

$$Q_{t|t-1} = \sigma^2 \frac{1 - e^{-\alpha \Delta t}}{\alpha} \left[ \beta \frac{1 - e^{-\alpha \Delta t}}{2} + e^{-\alpha \Delta t} \lambda_{t-1|t-1} \right] + e^{-2\alpha \Delta t} Q_{t-1|t-1}.$$

Observed and unobserved variables are linked by the so-called observation density $p(y_t|\lambda_t)$ that can be described as a measurement equation

$$y_t = A_t + B_t \lambda_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, h) \quad (31)$$

Thus, the expected values for the state space variables conditional on the information at time $t - 1$ are calculated as

$$\hat{y}_t = A_t + B_t \lambda_{t|t-1}.$$

The formulas for these two coefficients $A_t$ and $B_t$ can be derived from the pricing formulas for the hedge fund equity, an observable process in our case, as will be shown in the sequel.
Model without leverage constraint. In the first model we use Equation (6) to find the value of the equity when the intensity is given. We can transform this equation so that we get an observable process linearly depending on the intensity. We take the logarithm of both sides and get

\[
\ln(S(t)) = \ln(a) + \theta_0(\tau) - r\tau + \theta_1(\tau)\lambda(t) ,
\]

where \( \theta_0 \) and \( \theta_1 \) depend on the risk-neutral parameters \( \hat{\alpha}, \hat{\beta} \) and \( \sigma \). Hence, in terms of the parameters of the measurement equation (31), \( y_t = \ln(S(t)) \), \( A_t = \ln(a) + \theta_0(\tau) - r\tau \) and \( B_t = \theta_1(\tau) \).

Model with leverage constraint. For the second model the whole procedure is slightly more complicated because of the non-linear dependence between the observable and the hidden process. For this purpose we apply an extended Kalman filter and linearize Equation (29) by means of a first-order Taylor approximation around the best prediction \( \lambda_{t|t-1} \):

\[
S(t, \lambda(t)) = S(t, \lambda_{t|t-1}) + \frac{\partial}{\partial \lambda(t)} S(t, \lambda(t))|_{\lambda(t)=\lambda_{t|t-1}} \cdot (\lambda(t) - \lambda_{t|t-1}).
\]

(32)

Comparing equation (32) and the measurement equation (31), we define \( A_t \) and \( B_t \) as

\[
A_t = S(t, \lambda_{t|t-1}) - \frac{\partial}{\partial \lambda(t)} S(t, \lambda(t))|_{\lambda(t)=\lambda_{t|t-1}}
\]

and

\[
B_t = \frac{\partial}{\partial \lambda(t)} S(t, \lambda(t))|_{\lambda(t)=\lambda_{t|t-1}}.
\]

Thus, in both models we can calculate the state space variables \( \hat{y}_t \) based on the predicted values and formulas derived above. In the next step we can use an additional information provided by the observed variables \( y_t \) via the so-called correction step and update our estimations for \( \lambda(t) \).

\[
\lambda_{t|t} = max(\lambda_{t|t-1} + K \cdot [y_t - \hat{y}_t], 0)
\]

(33)

Alternatively CDS time series can be chosen as an observable process to estimate the parameters of the underlying process. We show the necessary calculations in Appendix 6.3.
and for the variance

\[ Q_{t|t} = [1 - KB_t]Q_{t|t-1}, \]

where \( K \) is the so-called Kalman gain matrix which is given as

\[ K = \frac{Q_{t|t-1}B_t}{Q_{t|t-1}B_t^2 + h}. \]

The update Equation (33) of the state estimate for the CIR model differs from the standard Kalman filter because of the non-negativity condition for the underlying process.

Finally, we calculate the quasi log-likelihood function for each parameter set \((\alpha, \beta, \sigma, \eta, a, b, h)\) for both models:

\[ \log L = -\frac{1}{2}(N - 1)\log(2\pi) - \frac{1}{2} \sum_{t=2}^{N} \log |F_t| - \frac{1}{2} \sum_{t=2}^{N} \frac{v_t^2}{F_t}, \]

with \( F_t = B_t^2Q_t + h \) and \( v_t = y_t - \hat{y}_t \) representing the error term. This function is further maximized with respect to the unknown parameters.

4 Empirical results

As an application of the models presented in the previous chapters we’ve chosen one of the former investment banks, Merrill Lynch, which we model as a hedge fund in our setup because of the type of its business. For a very long period the company was one of the leading and most renowned banks on Wall Street. However, due to the sub-prime crises, as many other banks, Merrill Lynch had enormous losses, in total about 40 billion USD. The company was even among the three banks that suffered most from the financial crisis. As a result, in September 2008, Merrill Lynch was taken over by the Bank of America. This acquisition seemed to be the only possibility for Merrill Lynch not to follow the tragic fate of Lehman Brothers which defaulted almost the same day.
4.1 Data

For our analysis we use the equity time series as well as a time series for 5 year CDS spreads provided by Bloomberg. Often stock prices rather than firm’s market capitalization are used. However, we preferred to work with firm values (stock price times number of shares outstanding) because in our view the profit or loss of an investor (changes in stock prices) is not as relevant for the entire creditworthiness of the company as its market capitalization. Also, this approach allows us not to take into account possible splits and changes in the firm’s capital structure.

Figure 1 shows the normalized values for market capitalization of Merrill Lynch in the period from January 2002 to July 2008 and the corresponding 5 year CDS spreads. Until the beginning of 2007 there were almost no signs for the coming crisis and the company’s equity was fluctuating around a positive trend. We can conclude the same from the second time series - decreasing values of 5 year CDS spreads. Since then we can see the negative dynamics in the firm’s value that became very strong after May 2007 accompanied by a dramatic increase of the CDS spreads. Changes in the firm’s credit quality are even more visible from dynamics of CDS spreads.

Considering both time series, the assumption of a negative correlation between credit spreads and firm’s equity can be assumed. The actual value of the correlation between those time series is $-51\%$ over the examined time period. In both cases we can see the influence of the financial market crisis - a tremendous worsening of the company’s financial and economic conditions in 2008, i.e. a rapid decrease in market capitalization and increasing CDS spreads which in this case can be seen as risk premium.

The examined period consists of 1639 daily observations. Moreover, to be consistent with the maturity of CDS contracts used in our analysis, we assume that the time to maturity of the option on the equity equals $T = 5$ years and represents the average maturity of Merrill Lynch’s debt. We also assume a flat interest rate at the level of 4%. Furthermore, we scale the observable equity time series by dividing it by the initial asset value $A(0)$. For the second model we assume a constant leverage of $X = D(0)/A(0)$. This information was extracted from the consolidated financial data of Merrill Lynch, as presented in Table 1 which shows some main figures characterizing the creditworthiness of the company and its over-

\footnote{Bloomberg ticker for the Merrill Lynch market capitalization is \texttt{MER US Equity} and for the 5 year CDS spreads - \texttt{MER CDS USD SR 5Y Currency}}

\footnote{Extracted from annual reports of Merrill Lynch for the corresponding years.}
4 Empirical results

Figure 1: Market capitalization and CDS spreads of Merrill Lynch (January 2002 - July 2008).

all economic and financial conditions for the time period between 2002 and 2008. This table justifies our assumption of constant leverage\(^8\) which we set to the initial value \(X = 0.9430\).

4.2 Application of the model

The equity time series in our model without leverage constraint is calculated via (6) with the following parameter set \(\Theta_1 = (\alpha, \beta, \sigma, \eta, a, b)\). Estimating these pa-

\(^8\) Fluctuations in leverage are less than 3%.
4.2 Application of the model

<table>
<thead>
<tr>
<th>Main figure (in million USD)</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
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<th>2007</th>
<th>2008</th>
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<td>EBIT</td>
<td>3,757</td>
<td>5,649</td>
<td>5,836</td>
<td>7,231</td>
<td>10,426</td>
<td>-12,831</td>
<td>-41,831</td>
</tr>
<tr>
<td>Net Revenues</td>
<td>18,608</td>
<td>20,154</td>
<td>22,023</td>
<td>26,009</td>
<td>34,659</td>
<td>11,250</td>
<td>-12,593</td>
</tr>
<tr>
<td>Total Assets</td>
<td>447,928</td>
<td>494,518</td>
<td>648,059</td>
<td>681,015</td>
<td>841,299</td>
<td>1,020,050</td>
<td>667,543</td>
</tr>
<tr>
<td>Total Liabilities</td>
<td>422,395</td>
<td>464,197</td>
<td>616,689</td>
<td>645,415</td>
<td>802,261</td>
<td>988,118</td>
<td>647,540</td>
</tr>
<tr>
<td>Stockholders’ Equity</td>
<td>22,875</td>
<td>27,651</td>
<td>31,370</td>
<td>35,600</td>
<td>39,038</td>
<td>31,932</td>
<td>20,003</td>
</tr>
<tr>
<td>Leverage</td>
<td>0.943</td>
<td>0.939</td>
<td>0.952</td>
<td>0.948</td>
<td>0.954</td>
<td>0.969</td>
<td>0.970</td>
</tr>
</tbody>
</table>

Table 1: Consolidated financial data of Merrill Lynch for 2002-2008

rameters via the Kalman filter method described in Section 3, we get

\[
\alpha = 0.3582 \quad (34)
\]

\[
\beta = 0.0012
\]

\[
\sigma = 0.0181
\]

\[
a = 2.0854
\]

\[
b = 1682.7
\]

\[
\eta = -0.2896.
\]

We use the same technique of the Kalman filter for the model with leverage constraint \( X \) that is set to the constant value of 0.9430 and get the following parameters

\[
\alpha = 0.4332 \quad (35)
\]

\[
\beta = 0.0021
\]

\[
\sigma = 0.0427
\]

\[
a = 1.2498
\]

\[
b = 1682.0
\]

\[
\eta = -0.1412.
\]

One of the biggest advantages of the Kalman filter is that, additional to the parameter estimations, it gives the values of the unknown state process, i.e. the intensity time series in our case. Knowing the intensity and the corresponding parameters it is possible to calculate the default probabilities by applying Formula (1). We use both parameter sets for our models and the corresponding intensity time series to calculate the default probabilities and compare them in Figure 2. We see similar evolutions of both time series. However, the default probabilities derived in the second model are slightly higher.
In a second step, we use the CDS time series to calculate the default probabilities inferred by the credit market and compare them to those derived from equity. The results are displayed in Figure 3. From this graph we can see that we get quite similar values for the default probabilities in the period from November 2002 until July 2007, when the credit market was relatively stable. However, in the times of high instability on the market, e.g. the current financial crisis, we get different values for the default probabilities. Despite the fact that both, the credit and the equity market, show the decline in creditworthiness, the default probabilities extracted from CDS spreads increase much faster and are much more volatile. Hence, the equity market doesn’t tell the whole story about the credit market and the other way around.

The difference between the two types of default probabilities can be explained by the so-called liquidity premium which we define as a difference between market quotes for CDS spreads and artificial CDS spreads implied by the equity. For a derivation of the CDS spreads see Appendix 6.3. We calculate two liquidity premium time series based on the model with (Second Model) and without (First Model) leverage constraint. However, as can be seen from the graph both lines
almost coincide during the entire period examined. This liquidity premium changes over time and represents the fluctuating structure of the credit market. As we can see from Figure 4, the liquidity premium in non-turbulent times is relatively small. However during some crises, e.g. the recent financial crisis or the burst of the dotcom bubble, the liquidity premium is higher than in stable periods. During this turbulent phases CDS spreads observed on the market are much higher and more volatile than those implied by the equity. Moreover, we can see that the liquidity premium is always positive, confirming the fact that stocks are still more liquid than the corresponding CDS.

5 Conclusion

In this paper we proposed two modeling approaches for the hedge fund’s equity within an intensity setup. In both cases the company’s equity is treated as a credit derivative with two different payoff structures - with and without leverage constraint. For the first model, the payment of the derivative only depends on
the corresponding value of the intensity. The second model takes into account the liabilities of the firm by including them into the payoff function. For both cases we were able to obtain analytical formulas based on the paper of Duffie et al. (2000) for the first model and Buettler, Waldvogel (1996) for the second one. We showed that the second model is a generalization of the first one and thus we provide another mechanism for deriving a closed-form solution for this type of affine models - Green’s function instead of ODEs. We used the Cox-Ingersoll-Ross process to describe the stochastic dynamics of the intensity. However, other common modeling frameworks as Vasicek (1977) or Hull-White (1990) are also applicable and represent possible directions for further research.

We applied both presented methods to the case of Merrill Lynch and used its equity time series to fit the parameters of the model. The estimation procedure is based on the extended Kalman filter and the method of least squares. We calculated the default probabilities and compared them to those inferred by the credit market, extracted from CDS spreads. We observed that in the imperturbable times the values of the default probabilities coincide. However, when the market is volatile and highly unstable the probabilities of default derived from CDS spreads are much higher than those implied by the equity. We explain this difference by means of a so-called liquidity premium.
6 Appendix

6.1 Note on the model without constraint

It is well known that the value of a zero-coupon bond with stochastic interest rate \( r \) is given by

\[
ZB(t,T) = \mathbb{E}\left[e^{-\int_t^T r(s) \, ds | F_t}\right].
\]

Moreover, there is a closed-form solution (see e.g. Brigo, Alfonsoi (2004)) when the underlying stochastic process follows a CIR process:

\[
ZB(t,T) = A(\tau)e^{-B(\tau) r(t)},
\]

where \( \tau = T - t \),

\[
A(\tau) = \left(\frac{2h e^{0.5(\alpha + h)\tau}}{2h + (\alpha + h)(e^{h\tau} - 1)}\right)^{2\alpha h \sigma^2}
\]

and

\[
B(\tau) = \frac{2(e^{h\tau} - 1)}{2h + (\alpha + h)(e^{h\tau} - 1)}.
\]

(36)

It can be shown that these formulas are a particular case of those derived in Section 2.1. There we got:

\[
\mathbb{E}[e^{-\int_t^T \lambda(s) \, ds} e^{-b\lambda(T) | F_t}] = e^{\theta_0(\tau) + \theta_1(\tau) \lambda(t)},
\]

(37)

where \( \theta_0 \) and \( \theta_1 \) were defined in Equations (8) and (9). Setting \( b = 0 \) and replacing \( \lambda \) by \( r \), we receive:

\[
\theta_1(\tau) = \frac{c(\alpha - h) + (\alpha + h)e^{-h\tau}}{\sigma^2(c + e^{-h\tau})},
\]

(38)

where

\[
c = \frac{h + \alpha}{h - \alpha}
\]

(39)

and

\[
h = \sqrt{\alpha^2 + 2\sigma^2}.
\]
Substituting (39) in (38) results in:

\[ \theta_1(\tau) = \frac{(\alpha + h)(1 - e^{-h\tau})(\alpha - h)}{\sigma^2(\alpha + h + e^{-h\tau}h - e^{-h\tau}\alpha)} \]

which after some transformations and using the fact that

\[ \alpha^2 - h^2 = -2\sigma^2 \]

we get:

\[ \theta_1(\tau) = -\frac{2(e^{h\tau} - 1)}{2h + (\alpha + h)(e^{h\tau} - 1)}. \]

The expression above coincides with \(-B(\tau)\) from (36). Analogously, for \(\theta_0(\tau)\), we first calculate the expression

\[ e^{\theta_0} = e^{\alpha\beta} \frac{2\alpha - h}{\sigma^2} \left( \frac{c + 1}{c + e^{-h\tau}} \right)^{\frac{2\alpha\beta}{\sigma^2}}, \]

which after some simple transformations gives

\[ e^{\theta_0} = \left( \frac{(c + 1)e^{0.5(\alpha + h)\tau}}{ce^{h\tau} + 1} \right)^{\frac{2\alpha\beta}{\sigma^2}}. \]

Substituting

\[ c = \frac{h + \alpha}{h - \alpha} \]

leads us to

\[ e^{\theta_0} = \left( \frac{2he^{0.5(\alpha + h)\tau}}{(2h + (\alpha + h)(e^{h\tau} - 1))} \right)^{\frac{2\alpha\beta}{\sigma^2}}. \]

Hence, \(e^{\theta_0(\tau)} = A(\tau)\) and, consequently, Formula (36) is a particular case of the generalized Formula (37).
6.2 Note on the model with constraint

As we have already mentioned in the Chapter 2, the closed-form solution from the second model with leverage constraint represents a generalized version of the first model. In the following we show that both analytical formulas coincide when $X = 0$. We recall the closed-form solution for the second model - Equation (30):

$$S(t) = a \cdot e^{-rt} P(\lambda, \tau) \left( \frac{p}{b + p} \right)^{k+1} e^{-\frac{t}{2} \frac{b}{p + b}}.$$  

We rewrite this equation using the formulas (16 - 25) in order to combine all constituents with and without $\lambda$:

$$S(t) = ae^{-rt} Q_0(\tau) e^{-Q_1(\tau)\lambda},$$

where

$$Q_0(\tau) = \left( \frac{2he^{(a+h)\tau/2}}{d_2} \right)^{2n\beta \sigma^2} \left( \frac{p}{p + b} \right)^{2n\beta \sigma^2}$$

and

$$Q_1(\tau) = \frac{4h^2 e^{h\tau}}{\sigma^2 d_2 d_1} \frac{b}{p + b} + \frac{2d_1}{d_2}.$$

For the first model without leverage constraint (Equation (6)) we have

$$S(t) = ae^{-rt} \hat{\theta}_0(\tau) e^{-\hat{\theta}_1(\tau)\lambda},$$

where, according to Equation (8),

$$\hat{\theta}_0(\tau) = e^{\hat{\theta}_0} = \left( \frac{(c + 1)e^{0.5(a+h)\tau}}{ce^{h\tau} + 1} \right)^{2n\beta \sigma^2},$$

and, according to Equation (9),

$$\hat{\theta}_1(\tau) = -\frac{ce^{h\tau}(\alpha - h) + (\alpha + h)}{\sigma^2(ce^{h\tau} + 1)}.$$

In order to verify the equivalence of both formulas for $S(t)$, we compare the parts including $\lambda$ and not-including $\lambda$. We start with the terms without $\lambda$. 

Substituting \( p = \frac{d_2}{\sigma d_1} \), we rewrite the expression for \( Q_0 \) as

\[
Q_0(\tau) = \left( \frac{2he^{0.5(\alpha + h)\tau}}{d_2} \cdot \frac{d_2}{\sigma^2 d_1 b + d_2} \right)^{2\alpha \beta \sigma^2}.
\]

Using \( d_1 = e^{h\tau} - 1 \), \( d_2 = (h + \alpha)e^{h\tau} + (h - \alpha) \), and \( c = \frac{h + \sigma^2 b + \alpha}{h - \sigma^2 b - \alpha} \) we further conclude that

\[
Q_0(\tau) = \left( \frac{2he^{0.5(\alpha + h)\tau}}{\sigma^2 d_1 b + d_2} \right)^{2\alpha \beta \sigma^2}.
\]

In the next step we compare the coefficients of \( \lambda \): \( \hat{\theta}_0 \) and \( Q_1 \). We start again from the one of the second model and use the fact, that \( p(\tau) = \frac{d_2}{\sigma^2 d_1} \):

\[
Q_1(\tau) = \frac{4h^2 e^{h\tau} b}{\sigma^2 d_1 d_2 b + d_2} + \frac{2d_1}{d_2}.
\]  

Using Equations (10), (11), (24), and (25), we can also simplify the expression for
6.3 Pricing of Credit Default Swaps

\[ \hat{\theta}_1 : \]

\[ \hat{\theta}_1(\tau) = -\frac{ce^{h\tau}(\alpha - h) + (\alpha + h)}{\sigma^2(ce^{h\tau} + 1)} \]

\[ = -\frac{(h + \sigma^2b + \alpha)e^{h\tau}(\alpha - h) + (\alpha + h)(h - \sigma^2b - \alpha)}{\sigma^2((h + \sigma^2b + \alpha)e^{h\tau} + (h - \sigma^2b - \alpha))} \]

\[ = -\frac{\sigma^2b(e^{h\tau}(\alpha - h) - (\alpha + h)) - (h^2 - \alpha^2)e^{h\tau} + (h^2 - \alpha^2)}{\sigma^2((h + \sigma^2b + \alpha)e^{h\tau} + (h - \sigma^2b - \alpha))} \]

\[ = -b(e^{h\tau}(\alpha - h) - (\alpha + h) + 2d_1) \]

\[ = \frac{d_2 + \sigma^2d_1b}{d_2(d_2 + \sigma^2d_1b)} \]

Now, using Equations (12), (24), and (25), we conclude

\[ -(e^{h\tau}(\alpha - h) - (\alpha + h))d_2 = (e^{h\tau}(h - \alpha) + (h + \alpha))((e^{h\tau}(h + \alpha) + (h - \alpha)) \]

\[ = e^{2h\tau}(h^2 - \alpha^2) + (h + \alpha)^2e^{h\tau} + e^{h\tau}(h - \alpha)^2 + (h^2 - \alpha^2) \]

\[ = 2e^{2h\tau}\sigma^2 + 4\alpha^2e^{h\tau} + 4\sigma^2e^{h\tau} + 2\sigma^2 \]

\[ = 4\alpha^2e^{h\tau} + 8\sigma^2e^{h\tau} + 2e^{2h\tau}\sigma^2 - 4\sigma^2e^{h\tau} + 2\sigma^2 \]

\[ = 4(\alpha^2 + 2\sigma^2)e^{h\tau} + 2(e^{h\tau} - 1)^2\sigma^2 \]

\[ = 4h^2e^{h\tau} + 2d_1^2\sigma^2 \]

and thus \( Q_1(\tau) = \hat{\theta}_1(\tau) \), i.e. we have shown that both expression for the hedge fund’s equity coincide, and consequently, the first model represents a particular case of the second model, when the leverage constraint is set to zero.

6.3 Pricing of Credit Default Swaps

A Credit Default Swap is a bilateral contract under which two counterparties agree to isolate and separately trade the credit risk of at least one third-party reference entity. The protection buyer pays a continuous premium, called the credit default swap spread, and receives from the protection seller a contingent payment upon a credit event. This payment in our set up is assumed to be equal to one minus the recovery rate \( R \), i.e. without loss of generality we set the notional to one. In the following we briefly present the derivation of the fair price of a CDS contract.
We discount all future payments using the flat interest rate \( r \geq 0 \) and assume as given the risk-neutral measure \( Q \). With \( \tau_D \) denoting the time of default of the reference entity, we obtain the following expression for the price of a contract with face value one, continuous premium payments \( S_{CDS} = S_{CDS}(0, T) \), and maturity \( T \):

\[
CDS(0, T) = \mathbb{E} \left[ e^{-r\tau_D} (1 - R) 1_{\{\tau_D \leq T\}} - \int_0^T S_{CDS} e^{-rt} 1_{\{\tau_D > t\}} dt \right] \\
= (1 - R) \int_0^T e^{-rt} dQ(\tau_D \leq t) - S_{CDS} \int_0^T e^{-rt} Q(\tau_D > t) dt. \quad (41)
\]

Market prices for CDS contracts are typically not quoted as in Equation (41). Instead, the spread which allows both parties to enter the contract at zero cost is quoted. This CDS spread is obtained from solving \( CDS(0, T) = 0 \) for \( S_{CDS} \), i.e.

\[
S_{CDS} = \frac{(1 - R) \int_0^T e^{-rt} dQ(\tau_D \leq t)}{\int_0^T e^{-rt} Q(\tau_D > t) dt}. \quad (42)
\]

Unfortunately it is not possible to invert this equation analytically in order to get the corresponding probability of default. Hence, we present a closed-form approximation for \( Q(\tau_D \leq t) \). We use the integration by parts formula to find

\[
\int_0^T e^{-rt} dQ(\tau_D \leq t) = e^{-rT} Q(\tau_D \leq T) + r \int_0^T Q(\tau_D \leq t)e^{-rt} dt \\
= e^{-rT} - e^{-rT} Q(\tau_D > T) + r \int_0^T e^{-rt} dt \\
- r \int_0^T Q(\tau_D > t)e^{-rt} dt \\
= 1 - e^{-rT} Q(\tau_D > T) - r \int_0^T Q(\tau_D > t)e^{-rt} dt.
\]

Inserting this formula into Equation (42), the CDS spread is given by

\[
S_{CDS} = (R - 1)r + \frac{(1 - R)(1 - e^{-rT} Q(\tau_D > T))}{\int_0^T e^{-rt} Q(\tau_D > t) dt}. \quad (43)
\]
Based on Equation (43), an upper boundary for the probability of default can be derived. For this we have \( Q(\tau_D > t) \leq 1 \) to receive

\[
S_{CDS}(0, T) \geq (R - 1)r + \frac{(1 - R)(1 - e^{-rT}Q(\tau_D > T))}{\int_0^T e^{-rt}dt}
\]
\[
= (R - 1)r + \frac{(1 - R)r(1 - e^{-rT} + e^{-rT}Q(\tau_D \leq T))}{1 - e^{-rT}}
\]
\[
= \frac{(1 - R)re^{-rT}Q(\tau_D \leq T)}{1 - e^{-rT}}.
\]

With \( R \in [0, 1) \) and \( e^{-rT} - 1 < 0 \) for \( rT > 0 \), the upper boundary for the probability of default is given by

\[
Q(\tau_D \leq T) \leq S_{CDS} \cdot \frac{(1 - e^{-rT})}{(1 - R)re^{-rT}}. \tag{44}
\]

**CDS spreads as observable process for Kalman filter**

As we have already mentioned in the Section 3, we can use the time series of CDS spreads in order to estimate the parameters of the CIR process \((\alpha, \beta, \sigma)\). Using Equation (12), the probability of default within our intensity setup is given by

\[
Q^D(t, T) = 1 - e^{\tilde{\theta}_0(\tau) + \tilde{\theta}_1(\tau)\lambda_t}
\]

On the other hand probabilities of default are inferred by CDS spreads and can be approximately extracted using the upper bound of Equation (44):

\[
Q^D(\tau) = S_{CDS}(\tau) \cdot \frac{(1 - e^{-r\tau})}{(1 - R)re^{-r\tau}}
\]

Hence,

\[
\ln \left( 1 - S_{CDS}(\tau) \frac{1 - e^{r\tau}}{(1 - R)re^{-r\tau}} \right) = \tilde{\theta}_0(\tau) + \tilde{\theta}_1(\tau)\lambda_t.
\]

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