Correlation Breakdown in the Valuation of Collateralized Fund Obligations.

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Abstract

In a collateralized fund obligation (CFO) several hedge funds are pooled in a fund that is in turn securitized. Since the first securitization of a fund of hedge funds, launched in June 2002, these products haven’t had much impact among investors. This is not surprising considering that it is very difficult to predict the credit solvency of each tranche of the securitization and therefore its valuation. Using hedge fund data, we analyze a particular CFO in an analytical pricing framework, capturing the correlation breakdown and the leptokurtic phenomena characteristic of hedge funds. We find that because of the lack of transparency characteristic of hedge funds, confidence intervals in probabilities of default and credit spreads, are high enough to burden CFO proliferation.
Collateralized Debt Obligations (CDO) are structured credit vehicles that redistribute credit risk to meet investor demands for a wide range of rated securities with scheduled interest and principal payments. CDOs are securitized by diversified pools of debt instruments. Recent developments in credit structuring technology include the introduction of Collateralized Fund Obligations (CFO). The capital structure of a Collateralized Fund Obligation is similar to traditional CDOs, meaning that investors are offered a spectrum of rated debt securities and equity interest. Although any managed fund can be the source of collateral, the target collateral in these structures tends to be hedge funds, such as relative value hedge funds, event-driven hedge funds or commodity trading advisors (CTAs), along with funds that finance the needs of growing companies, such as private equity and mezzanine funds. Often, a special purpose entity purchases the pool of underlying hedge fund investments, which are then used as collateral to back the notes.

These asset-backed notes are also called tranches. The most senior tranche is usually rated AAA and is credit-enhanced due to the subordination of lower tranches. This means that the lowest tranche, which is typically the equity tranche, absorbs losses first. When the equity tranche is exhausted, the next lowest tranche begins absorbing losses. A CFO may have an AAA-rated tranche, an AA-rated tranche, a single A tranche, a BBB rated tranche and an equity tranche. Exhibit 1 shows a schematic prototype of a CFO structure. Therefore, a CFO can be regarded as a financial structure with equity investors and lenders where all the assets, equity and bonds, are invested in a portfolio of hedge funds. The lenders earn a spread over interest rates and the equity holders, usually the manager of the CFO, earn the total return of the fund minus the financing fees.

As pointed out in recent articles (Mahadevan and Schwartz [2002] and Stone and Zissu [2004]) on this topic, both investors and issuers find CFO securizations attractive. Investors, because a triple-A-rated Bond will have a similar yield to that of a triple-A collateralized debt obligation plus a premium because with this structure they gain exposure to a diverse collection of hedge funds through a fund of funds manager. And issuers, because securitization is a convenient vehicle for raising funds for an otherwise relatively illiquid product. Stone and Zissu [2004] provide a detailed overview of the first securitization of a fund of funds, the Diversified Strategies CFO SA, launched in June 2002. This CFO issued US $251 million in five tranches, rated AAA, AA, A, BBB and the equity. Also, in June of 2002, lawyers from the Structured
Products Group completed the second Collateralized Fund Obligation backed by hedge fund portfolios and assigned ratings by Moody's Investors Service, Inc. and Standard & Poor's Rating Services. The CFO, titled Man Glenwood Alternative Strategies I or "MAST I" issued rated notes totaling US $374 million and nonrated notes and preference shares totaling U.S. $176 million. As the first of their kind, these CFOs were viewed as a cutting edge transaction within the industry and attracted considerable attention in the financial press.

As far as valuation is concerned, Moodys in August 2003 began to use HedgeFund.net data to evaluate the risks of the underlying collateral in order to develop an accurate Montecarlo based rating model for CFOs (see Moodys [2003]). Although CDOs have attracted much attention in the academic literature (Hull and White [2004], Li [2000], Laurent and Gregory [2003]), no effort has been made to develop an analytical rating model for CFOs.

The credit rating of a CFO tranche depends directly on the mark-to-market value of the pool of hedge funds. Collateralized fund obligations tend to be structured as arbitrage market value CDOs, meaning that the fund of funds manager focuses efforts on actively managing the fund to maximize total return while restraining price volatility within the guidelines of the structure. The diversity of hedge fund investment strategies and the active management of portfolio positions among strategies ensure that hedge fund risk and return characteristics are different from those of traditional assets (equities, bonds or ETFs) as illustrated by numerous articles in the literature (e.g. Moodys [2003], CISDM [2006]). The unique return and risk characteristics of hedge funds are better characterized by more general distributions than the usual multivariate Gaussian approach. Therefore the probability of default of the different tranches will be also more volatile. In addition lack of transparency within hedge funds in general, makes it more difficult to obtain high-frequency historical data which leaves investors with no choice but to calibrate rather complex valuation models including the typical non-gaussian behavior of hedge funds with monthly data.

With the ultimate aim of assessing the model and calibration risk incurred on by CFOs, we develop in this article an analytical procedure for valuing the different tranches of a CFO by calculating the probability of default and the credit spreads from the bonds and analyzing the pricing sensitivities to changes in parameters. When calculating the probabilities of default and the credit spreads of the different tranches of a CFO one should take into consideration the non-gaussian behavior of the returns from the pool of hedge funds. As we shall show later, those
credit spreads are directly related to the percentiles of the underlying portfolio distribution.

In our pricing model we will assume that returns from the pool of hedge funds follow a Covariance-Switching (CS) Stochastic Process which is defined in the Appendix. This framework allow us to incorporate two well known characteristics from the return series of hedge funds: first, the skewed and leptokurtic nature of the marginal distribution functions and second the asymmetric correlation or correlation breakdown phenomenon (Longin and Solnik [2001]). As we will prove later, the correlation between the different hedge funds depends on the direction of the market. For instance, correlations tend to be larger in a bear market than in a bull market.

The structure of this article is as follows. Providing empirical evidence of the existence of correlation regimes, in Section 2 we will present the Covariance-Switching process as the model for the profit and loss function for a pool of hedge funds and will outline some of their properties. The model for valuing a CFO will be presented in Section 3 and in Section 4 we will illustrate the strong parameter dependence with an example of a hypothetical CFO based on the S&P CTA index. This methodology can also be applied to other daily hedge fund indexes such as the Dow Jones Hedge Fund Strategy Benchmarks and others. Section 5 contains our conclusions.

Pool of Hedge Funds Model.

Hedge funds in general follow dynamic investment strategies using a wide variety of assets. Typically the investment processes is not transparent and in most cases knowledge about future cash flows to be generated by the fund may be closely guarded. Hedge funds are largely unregulated because they are typically limited partnerships with fewer than 100 investors, which exempts them from the Investment Company Act of 1940.2 Offshore hedge funds are non-U.S. corporations and are not subject to SEC regulation. This limited regulation allows hedge funds to be extremely flexible in their investment options. Hedge funds can use short selling, leverage, derivatives, and highly concentrated investment positions to enhance returns or reduce systematic risk. They can also attempt to time the market by moving quickly across diverse asset categories. Hedge funds attract mainly institutions and wealthy individual investors,
with minimum investments typically ranging from $250,000 to $1 million. Additionally, hedge funds often limit an investor’s liquidity with lock-up periods of one year for initial investors and subsequent restrictions on withdrawals to certain intervals. Specific investments made by hedge funds are often carefully guarded secrets. This lack of transparency may make it difficult for a fund of hedge funds manager to assess the fund’s aggregate exposure to a particular investment on a portfolio basis. It also presents a challenge to the fund manager attempting to monitor a particular hedge fund’s adherence to its advertised style or investment approach. However in certain cases the transparency issues indicated above can be alleviated if the manager allows separate accounts. In order to account for the plethora of hedge fund and CTA strategies and the speed at which a given fund’s composition can be changed, we model the alternative assets to exhibit more exotic distributions than the Gaussian one, displaying asymmetric and leptokurtic returns.

As is the case with most diversified portfolios, fund of hedge funds reduce the numerous types of risks arising from individual hedge funds by diversifying both across strategies as well as the number of funds in each strategy. Both types of diversification reduce risk, but because of the potential for style drift and a rapid total loss of value in a single fund, diversification across funds is especially important. Diversification is most effective for assets that exhibit low correlations in value over time. Fund of hedge funds’ managers pursuing a low-volatility strategy seek to assemble a portfolio of relatively uncorrelated assets. Unfortunately, in times of market distress, such as the second half of 1998, previously uncorrelated hedge funds or CTAs can become highly correlated for short periods. This phenomenon is often called correlation breakdown. Evidence of this is presented in Exhibit 2 which shows the correlation matrices between the return series of hedge fund managers under tranquil and distressed regimes. During tranquil periods correlations are lower, whereas during periods of market distress, the asset returns become highly correlated, with the magnitudes of off-diagonal correlation values being close to one in absolute terms. Therefore diversifying amongst different assets or markets in times of market distress was less effective at reducing risk than many participants had hitherto believed. Articles exploring the contagion phenomenon include Harvey and Viskanta [1994], Longin and Solnik [2001] and Koedijk and Campbell [2002].

In order to capture both the unidimensional leptokurtic and asymmetric nature of hedge fund returns and the correlation breakdown phenomenon, we select the Covariance-Switching...
Stochastic Process (CS process) for modelling the returns of a pool of hedge funds from the range of parametric alternatives to a Geometric Brownian Process. With this election we develop a very tractable model (calculations using it often closely resemble those using a Brownian Process) and allow for a good adjustment to market data. Regime switching models have been used before in the field of finance but mainly in its univariate version, see Yao, Zhang and Zhou [2003] for option pricing and Choi [2004] for interest rate modelling. Moreover regime switching models have the theoretical appeal that by adding together a sufficient number of components, any multivariate distribution may be approximated with reasonable accuracy. With an infinite number of contributions, any distribution can be reconstructed exactly.

A general CS is defined through its stochastic differential equation as follows:

**Definition:** $X_i(t)$ follow a covariance switching process with parameters $(p, \lambda, \mu, \mu_i^0, \mu_i^1, \sigma_i^0, \sigma_i^1)$ if the diffusion process can be represented as:

$$
\begin{align*}
  dX_i(t) &= \mu_i(t)dt + \sigma_i(t) \cdot dW(t) \\
  \mu_i(t) &= \mu + J_t \cdot \mu_i^0 + (1 - J_t) \cdot \mu_i^1 \\
  \sigma_i(t) &= J_t \cdot \sigma_i^0 + (1 - J_t) \cdot \sigma_i^1 \\
  \sigma_i^k(t) &= (\sigma_i^{1,k}(t), ..., \sigma_i^{n,k}(t))
\end{align*}
$$

Where $J_t$ is a jump process (see Appendix for details), $i = 1, ..., n$, $j = 1, ..., d$, $W_t$ is a $n$-dimensional vector of independent Brownian motion processes, which are independent of $J_t$. Moreover, $\sigma_i^{j,k}, \mu_i^k$ are constants ($k = 0, 1; i = 1, ..., n; j = 1, ..., n$).

In this work the vector of hedgefunds will be assumed $CS(p, \lambda, \mu, \sigma^{-0}, \sigma^{-1})$ under the historical measure ($P$-measure). Notice no jump in the drift is assumed, and then they are $CS(p, \lambda, r, \sigma^{-0}, \sigma^{-1})$ under the risk neutral measure ($Q$-measure). Therefore the returns will follow $CS(p, \lambda, \mu, \mu^{-0}, \mu^{-1}, \sigma^{-0}, \sigma^{-1})$ under $P$ and $CS(p, \lambda, \mu^{-0}, \mu^{-1}, \sigma^{-0}, \sigma^{-1})$ under $Q$ (see Appendix for details).

Notice that the conditional distribution implied from a Covariance-Switching process is not closed form but a discrete approximation that leads to a gaussian mixture distribution, providing an intuitive interpretation of CS. For example, taking an increment $\Delta t = 1$ then a
discrete approximation to the CS density would be the density function of a mixture of two multivariate gaussians:

$$f(x) = \frac{p (1 - e^{-\lambda})}{\sqrt{2\pi \det \Sigma_0}} e^{-\frac{1}{2}(x-\mu_0)^t \Sigma_0^{-1} (x-\mu_0)} + \frac{1 - p (1 - e^{-\lambda})}{\sqrt{2\pi \det \Sigma_1}} e^{-\frac{1}{2}(x-\mu_1)^t \Sigma_1^{-1} (x-\mu_1)}$$

where $\mu_i = \mu + \mu^i$ and $\Sigma_i = \sigma^{\cdot i} \cdot \sigma^{\cdot i}$ are the $i$-th means vector and the $i$-th variance-covariance matrix. If the given Gaussian mixture GM distribution were used to describe the monthly returns of a portfolio, with the first gaussian one could model the tranquil regimes and with the other the distressed ones and moreover, the parameter $p$ could be interpreted as the probability of having a tranquil month while $1 - p$ would be the probability of a distressed month. Unfortunately, GM is not the conditional distribution of a CS process (see Appendix), the main difference between GM and the conditional distributions of CS being that for CS $p$ would be a function of time whose path affects the gaussianity of the components in the mixture.

Fitting parameters to a Covariance-Switching Process has not been explored in great detail in the literature. Few papers address the unidimensional case, see Choi [2004], Chourdakis [2002] but none, as far as we known, deals with multidimensional situations. Therefore we use an ad-hoc algorithm based on a multivariate test of gaussianity. The algorithm focuses on detecting the moments where a jump has occurred, and then uses the samples from tranquil and distress scenarios to estimate standard multidimensional gaussian processes. As a first pass, we define a distressed month as a month where any of the returns of the portfolios’ hedge funds is greater in absolute value than two standard deviations and a tranquil month when all of them are smaller, then we add or remove sample points in order to create two groups where the hypothesis of multidimensional gaussianity is accepted.

**CFO Model**

In our proposed model we will make the following assumptions. First, we consider a single period model, typically a year, in which distributions of random variables are sufficient to specify the model. Furthermore we will suppose that returns of the pool of hedge funds follow a Covariance-Switching process $CS(p, \lambda, r, \mu^{0}, \mu^{1}, \sigma^{0}, \sigma^{1})$. Second, we assume that there is no risk of default in the pool of hedge funds, therefore the portfolio’s profit and loss function will be only market driven. This assumption will be reasonable as far as the fund of hedge funds doesn’t contain a high proportion of speculative grade hedge funds in order that that the
probability of default of each hedge fund is not relevant. Besides, introducing more complexity in the model will result in an increase in the number of parameters and even with the simple model that we propose we already have large confidence intervals as we’ll see later.

One could adopt the more realistic view but rather abstract view that many hedge funds are nothing but a junior tranche of a massive credit derivative, in recognition of their propensity to default (as was the case of Long Term Capital Management, Beacon Hill, and more recently, Amaranth, among many others). From this viewpoint, then the tranche of a CFO on a fund of hedge funds will be come a second order credit derivative, analog of a CDO squared. The analysis of this, to which our discussion here will provide a frame of reference, is left to a later study. However, this second derivative nature of a fund of funds CFO only hightens the need to study non-gaussian dependence behavior among the hedge funds, as dependence will not behave linearly in the tails of the distribution.

Let us introduce some notation:

- $S_0^i$ and $S_i$ are the initial and end values of the $i$-th hedge fund in the pool, $1 < i < n$.
- $\pi$ and $\pi_0$ are the initial and end values of the portfolio.
- $R_i$ is the return of the $i$-th hedge fund.
- $n_i$ is the number of units invested in the $i$-th hedge fund and $\theta_i$ is the monetary quantity invested, so that $\theta_i = n_i S_0^i$.
- $PL = \pi - \pi_0$ is the profit and loss of the portfolio and $R_{PL} = \frac{\pi - \pi_0}{\pi_0}$ is the relative profit and loss.
- $I_m$ is the quantity invested by tranche $m$, $1 \leq m \leq M$. And $D_m + 1 = \sum_{i=1}^{m-1} \frac{I_i}{\pi_0}$ is the relative cumulative quantity invested, $D_{M+1} = 0$.
- The default rate $r^*$ is given by $L_m = I_m e^{r^*}$ where $L_m$ is the undertaken future cashflows in one year of tranche $m$:

$$L_m = \begin{cases} I_m (1 + R_{PL}) & \text{if } R_{PL} > 0 \\ I_m & \text{if } D_{m+1} < R_{PL} < 0 \\ I_m \left( \frac{R_{PL} \cdot D_m}{D_{m+1} \cdot D_m} \right) & \text{if } D_m < R_{PL} < D_{m+1} \\ 0 & \text{if } R_{PL} < D_m \end{cases}$$  \tag{3}
The recovery rate, $f_m$ of tranche $m$, would be $\frac{R_{PL} - D_m}{D_{m+1} - D_m}$.

- $r$ is the risk free rate.

- $s_m = r^* - r$ is the default spread.

Consider a portfolio $\pi$ with underlying hedge funds $S_1, ..., S_n$, each with $n_i$ positions. It's mark-to-market is given by:

$$\pi = \sum_{i=1}^{n} n_i S_i$$

(4)

and it’s profit and loss return is given by:

$$PL = \pi - \pi_0 = \sum_{i=1}^{n} \theta_i \left( \frac{S_i - S_0^0}{S_0^0} \right) = \sum_{i=1}^{n} n_i R_i$$

(5)

Then the tranche $m$ will drop in default if:

$$\pi - \pi_0 = PL < \sum_{i=1}^{m-1} I_i - \pi_0$$

That is, if the mark-to-market of the portfolio at the end of the period is less than the invested quantity of the $m-1$ tranche of less seniority then the $m$-th will default. We can express this inequality in terms of the relative profit and loss and the relative weights in each tranche:

$$R_{PL} = \frac{\pi - \pi_0}{\pi_0} < \sum_{i=1}^{m-1} \frac{I_i}{\pi_0} - 1 = D_m$$

Note that $D_1 = 0$. Therefore the probability of default from the $m$-th tranche, $PD_m$, equals to:

$$PD_m = P \left( R_{PL} < D_{m+1} \right)$$

(6)

This equation relates the probability of default to the profit and loss distribution function of the underlying pool of hedge funds. Basically the one year probability of default is related to the one year VaR of the collateral portfolio. Under the assumption that the returns of the different hedge funds, $R_i$, follow a unidimensional CS process $CS(p, \lambda, r, \mu_i^0, \mu_i^1, \sigma_i^0, \sigma_i^1)$, using a result presented in appendix, the profit and loss of the portfolio follows itself a unidimensional CS process with parameters $CS(p, \lambda, \sum_{i=1}^{d} \theta_i \mu, \theta \cdot \mu^0, \theta \cdot \mu^1, \theta \cdot \sigma^0 \cdot \sigma^0 \cdot \theta', \theta \cdot \sigma^1 \cdot \sigma^1 \cdot \theta')$. Hence, in
order to calculate the probabilities of default of the different tranche, we only need to calculate
the percentiles of a unidimensional CS process. Exhibit 3 presents a schematic picture of the
model.

Moreover we can obtain the distribution function of the recovery rate of the tranche \( m \), \( f_m \),
as:

\[
P(f_m < x) = P \left( \frac{R_{PL} - D_m}{D_{m+1} - D_m} < x \mid D_m < R_{PL} < D_{m+1} \right)
\]

\[
= \frac{P(D_m < R_{PL} < x(D_{m+1} - D_m) + D_m)}{P(D_m < R_{PL} < D_{m+1})}
\]

(7) (8)

Subsequently, we can compute the spread of the \( m \)-th tranche as the discounted expected
value of the one year cashflow \( L_m \) under the risk-neutral probability \( Q \):

\[
I_m = L_m e^{-r^*_m}
\]

\[
s_m = r^*_m - r
= \ln \left( E^Q \left[ (1 + R_{PL}) 1_{\{R_{PL}>0\}} + 1_{\{D_{m+1}<R_{PL}<0\}} \right]
\right.

\left. + \left( \frac{R_{PL} - D_m}{D_{m+1} - D_m} 1_{\{D_m<R_{PL}<D_{m+1}\}} \right) \right)
\]

(9) (10)

We have to take into account that the yield spread between a corporate bond and an otherwise
identical bond with no credit risk reflects the expected actuarial loss, or annual expected loss
given default, plus a risk premia reflecting for example liquidity risk. It is possible to calculate
the probability of default and the spread of tranche \( m \), details are given in the Appendix.

**A sample CFO**

As explained before, CFOs are products which haven’t had too much impact among in-
estors. This fact, may be explained due to huge variability of the probability of default with
respect to market conditions. To illustrate this feature, we will analyze a CFO with the fol-
lowing characteristics: There is a total investment of $1 million dollars distributed within five
tranche with weights 0.4976, 0.13, 0.0398, 0.065 and 0.2676 respectively. The tranche with the
26% is the equity tranche and the tranche with the 49% is the typically AAA rated tranche.
These weights are in agreement with market conventions and were used in the first CFO created, Diversified Strategies CFO SA DSCFO1. With this assumptions the limiting losses would be equal to $D_6 = 0$, $D_5 = -0.2676$, $D_4 = -0.3326$, $D_3 = -0.3724$ and $D_2 = -0.5024$.

The pool of Hedge Funds is chosen to be the following 6 Hedgefunds: BDC Offshore Fund Ltd (A), Kassirer Market Neutral, Mapleridge Fund LP, New Ellington Overseas, Recap Inter and Styx International Ltd, we work with monthly data returns from jan-2001 to oct-2005. The investment in each hedge fund is equal to $1/6$M$\$. 

First we estimate the parameters of our model using the method proposed in the appendix. Notice that hedge funds report monthly returns instead of daily leading to a lack of data. Our method is effective because it reduces the number of parameters, allowing for more reliable estimations and accurate intervals for the probabilities of default or for the spread calculations favoring the bid and offer pricing process.

In exhibit 4 we present the estimations of the variances for all CTAs returns in the distressed and tranquil regimes and the mean under the historical measure. Variances are greater in distressed than in tranquil regimes. In exhibit 5 we present the estimated correlations in the tranquil and distressed regime. One can observe the correlation breakdown phenomenon. Finally the proportion of tranquil months converges to $0.844$ as $t$ approaches infinity.

Using these estimations we can calculate the parameters of the monthly relative profit and loss portfolio returns under the P-measure. Results are $\mu = 0.101$, $\sigma^0 = 0.042$, $\sigma^1 = 0.089$, $p = 0.867$, $\lambda = 1.029$. In order to get the yearly returns distribution one needs to multiply monthly means and variances by twelve. With this results, using simulations and equation (2) we are able to calculate the probabilities of default of the different tranche. In exhibit 6 we can see that the equity tranche has a one year probability of default of $19.5\%$ and the most senior tranche a probability of default of $0.03\%$. Moreover we report the spread over the risk free rate, supposed to be $1\%$. Therefore a mezzanine tranche investor should be expecting a spread of $2.3\%$ and the most senior tranche investor should be given and spread of $4.85$ basic points. We do not report a spread value for the equity tranche because usually the yearly coupon on this tranche depends on the final portfolio value.

**Parameter Sensitivities**

Up to now, we have calculated the probabilities of default of the different tranche from the
CFO according to market conditions during the period of Jan-2001 to Oct-2005. However, how would have changed those probabilities if market conditions would have been slightly different? In order to answer that question we will recalculate the probabilities of default changing the probability \( p \) of a jump in equation (1). In this way we are measuring the effect in default probabilities of having more probability of being in a distressed regimes. Therefore we evaluate the CFO for a grid of probabilities in the interval going from 0 to 1. Exhibit 7 shows that the probabilities of default spread over a substantial range when changing the probability of a distress month \( 1-p \) (market conditions). For example the mezzanine tranche probability of default could go from 2% to 9%. In Exhibit 8 we report the sensitivities of the spread yield to the market condition parameter \( p \) which present a similar behavior to probabilities of default. In both cases a confidence interval for \( p \) could be attractive as a reliability measure. A good proxy for the confidence interval of the parameter \( p \) can be obtained using the binomial distribution, for example the 95% confidence interval for \( p \) ranges from 0.725 to 0.926 therefore that the spread confidence interval is (1.9%, 3.8%). As a consequence investors could react with foresight when investing in this type of collateralized fund obligations.

**Conclusions**

A covariance switching multidimensional process was proposed and studied for the pricing of collateralized fund obligations (CFO). This process presents several desirable properties, such as closeness under linear transformation. This implies, in particular, that marginals and portfolios will belong to the CS family of processes. Another interesting property of this process is its simplicity for providing cumulative distribution probabilities in the unidimensional case, which enables us to compute portfolio cumulative probabilities by simply simulating a jump process. A method to estimate the parameters was also proposed. An empirical analysis shows a better than gaussian fitting to a time series vector of hedgefunds; the large differences for the covariance matrices in tranquil and distress periods not only support the stylized facts described in the literature regarding leptokurtic unidimensional behaviors but also the reported correlation breakdown. We find that because of the lack of transparency characteristic of hedge funds as well as the unavailability of data, confidence intervals in probabilities of default and credit spreads are high enough to burden CFO proliferation. For example we find a 95% credit
spread interval of 1.9%-3.8%.

References


Appendix: Covariance-Switching Process.

A Covariance-Switching process will be presented in this section.

Let’s first define a jump process \( J_t \in \{0, 1\} \). \( P[\Delta J_t \in -1 \mid J_t^- = 1] = d(t), P[\Delta J_t \in 1 \mid J_t^- = 0] = u(t) \).

The law of a jump \( J_t \) is described by:

- The intensity of jumps: \( \lambda(t) dt \)
- The law of the jumps: \( K(t, dy) = P[\Delta J_t \in dy \mid J_t^-] \).

With previous notation, it follows:

\[
K(t, dy) = P[\Delta J_t \in dy \mid J_t^-] = J_{t-} \left[ (1 - d(t))\delta_0 + d(t)\delta_{u(t)} \right] (dy) + (1 - J_{t-}) \left[ (1 - u(t))\delta_0 + u(t)\delta_{u(t)} \right] (dy)
\]

, where \( \delta_x(dy) \) denotes the dirac delta. It is known that \( J_t = J_0 + \int_0^t \int_E y\lambda(s)K(s, dy)ds + J_t^d \), where \( J_t^d \) is a local martingale (purely discontinuous martingale) such that \( J_0^d = 0 \) and \( d\langle J_t^d, M \rangle = 0 \) for any continuous local martingale \( M \).

Remark: Notice that \( \int_E y\lambda(s)K(s, dy) = J_{s-} (-d(s)) + (1 - J_{s-})u(s) \). Then

\[
J_t = J_0 + \int_0^t (J_{s-} (-\lambda(s)d(s)) + (1 - J_{s-})\lambda(s)u(s)) ds + J_t^d
\]

\[
J_t = J_0 + \int_0^t \lambda(s) [u(s) - (d(s) + u(s))J_{s-}] ds + J_t^d
\]

Moreover.

\[
E [J_t \mid J_0] = J_0 + \int_0^t \lambda(s) (u(s) - (d(s) + u(s))E [J_{s-} \mid J_0]) ds
\]

Let us called \( q(t) = E [J_t \mid J_0] \) then:

\[
q(t) = \left[ J_0 + \int_0^t \lambda(s) u(s)e^{\int_0^s \lambda(r)(u(r) + d(r))dr}ds \right] e^{-\int_0^t \lambda(s)(u(s) + d(s))ds}
\]

(11)
Notice that \( q(t) \) provides the probability of \( J_t = 1 \) given information up to \( t = 0 \). For simplicity, we assume the following parameters, \( \lambda(t) = \lambda, \ u(t) = p, \ d(t) = 1 - p \), which leads to:

\[
q(t) = J_0. e^{-\lambda t} + p \left( 1 - e^{-\lambda t} \right) \quad (12)
\]

Definition: \( X_i(t) \) follow a covariance switching process with parameters \((p, \lambda, \mu_i^0, \mu_i^1, \sigma_i^{0}, \sigma_i^{1})\) if the diffusion process can be represented as:

\[
\begin{align*}
dX_i(t) &= \mu_i(t)dt + \sigma_i(t) \cdot dW(t) \\
\mu_i(t) &= \mu + J_t \cdot \mu_i^0 + (1 - J_t) \cdot \mu_i^1 \\
\sigma_i(t) &= J_t \cdot \sigma_i^{0} + (1 - J_t) \cdot \sigma_i^{1} \\
\sigma_i^{j,k}(t) &= \left( \sigma_i^{1,k}(t), \ldots, \sigma_i^{n,k}(t) \right)
\end{align*}
\]

Where \( J_t \) is a jump process defined previously, \( i = 1, \ldots, n, j = 1, \ldots, d \), \( W_t \) is a \( n \)-dimensional vector of independent Brownian motion processes, which are independent of \( J_t \). Moreover, \( \sigma_i^{j,k}, \mu_i^k \) are constants \((k = 0, 1; i = 1, \ldots, n; j = 1, \ldots, n)\).

Property 1: If \( P(t) = \sum_{i=1}^{d} a_i \cdot X_i(t) \), where \( X_i \) follows a CS process with parameters \((p, \lambda, \mu_i^0, \mu_i^1, \sigma_i^{0}, \sigma_i^{1})\) then \( P(t) \) follows a covariance-switching process with parameters \((p, \lambda, \sum_{i=1}^{d} a_i \mu_i, a \cdot \mu^0, a \cdot \mu^1, a \cdot \sigma^0 \cdot \sigma^{0'} \cdot a', a \cdot \sigma^1 \cdot \sigma^{1'} \cdot a')\).

Process for the underlying hedgefunds.

\( S_i(t) \) follow a covariance switching process with parameters \((p, \lambda, \mu_i, \sigma_i^{0}, \sigma_i^{1})\):

\[
\begin{align*}
\frac{dS_i(t)}{S_i(t)} &= \mu_i dt + \sigma_i(t) \cdot dW^P(t) \\
\sigma_i(t) &= J_t \cdot \sigma_i^{0} + (1 - J_t) \cdot \sigma_i^{1} \\
\sigma_i^{j,k}(t) &= \left( \sigma_i^{1,k}(t), \ldots, \sigma_i^{n,k}(t) \right)
\end{align*}
\]

Where \( P \)-historical measure, \( J_t \) is the jump process defined previously, \( i = 1, \ldots, n, j = 1, \ldots, d \), \( W_t \) is a \( n \)-dimensional vector of independent Brownian motion processes, which are independent of \( J_t \). Moreover, \( \sigma_i^{j,k} \) are constants \((k = 0, 1; i = 1, \ldots, n; j = 1, \ldots, n)\).
We assume that the jump process does not change with a change of measure (see Merton [1974] for an explanation of the plausibility of this assumption) therefore if $Q$ is the risk free measure then:

\[
\frac{dS_i(t)}{S_i(t)} = r dt + \sigma_i(t) \cdot dW^Q(t)
\]

Property 2: Assumes $X_i(t) = \ln S_i(t)$ then, by Ito’s lemma, $X_i(t) = rt - \frac{1}{2} \int_0^t \sigma_i(s) \cdot \sigma_i(s)’ ds + \int_0^t \sigma_i(s) \cdot dW^Q(s)$.

Remark: The distribution of $X(t) = (X_1(t), ..., X_d(t))$ conditional on the history of $J_s$ for $0 \leq s \leq t$, under the Q-measure, is multivariate Gaussian with mean and volatility as follow ($t_j$ denotes the time of a jump, $n$-number of jumps, both known under the assumption):

\[
\mu = rt - \sum_{j=1}^n \left[ \sum_{l=1}^d \left( \sigma_{ij}^{lk} \right)^2 \cdot (t_j - t_{j-1}) \right]
\]

\[
\sigma = \sum_{j=1}^n \left[ \sum_{l=1}^d \left( \sigma_{ij}^{lk} \right) \left( \sigma_{kj}^{lk} \right) \cdot (t_j - t_{j-1}) \right]
\]

\[
k = \sin^2 \left( \frac{\pi j}{2} \right)
\]

For example:

\[
\begin{cases}
Nd \left( \mu^{(1)}, \Sigma^{(1)} \right) & \text{if } J_s = 1, 0 \leq s \leq t \\
Nd \left( \mu^{(2)}, \Sigma^{(2)} \right) & \text{if } J_s = 0, 0 \leq s \leq t
\end{cases}
\]

where the $i$ component of $\mu^{(k)}$ ($k = 0, 1$) is $(r - \sum_{l=1}^d \left( \sigma_{ij}^{lk} \right)^2) \cdot t$, while the $i, j$ component of $\Sigma^{(k)}$ is $\sum_{l=1}^d \left( \sigma_{ij}^{lk} \right) \left( \sigma_{kj}^{lk} \right) \cdot t$.

These features suggest that a change in $J_t$ can be seen as a change in the market conditions, leading to a change in trend not only for the volatility of the hedgefunds but also the correlation among them.

Property 3: Assumes $\Pi(t) = \sum_{i=1}^d a_i \cdot X_i(t)$, $\sum_{i=1}^d a_i = 1$, then, by Ito’s lemma, $d\Pi(t) = \left[ r - \frac{1}{2} \sum_{i=1}^d a_i \sigma_i(t) \cdot \sigma_i(t)' \right] dt + \left[ \sum_{i=1}^d a_i \sigma_i(t) \right] \cdot dW^Q(t)$. Moreover, the distribution of $\Pi(t)$ condi-
tional on observing the history of $J_t$ is normal with the following mean and volatility:

$$
\mu_T(J) = rt - \sum_{i=1}^{d} a_i \left[ \sum_{j=1}^{d} \left\{ (\sigma_i^{j,0})^2 F_i + (\sigma_i^{j,1})^2 (t - F_i) \right\} \right]
$$

$$
= rt - \sum_{i=1}^{d} a_i \left[ \sum_{j=1}^{d} (\sigma_i^{j,1})^2 t \right] + \left[ \sum_{i=1}^{d} \left( \sum_{j=1}^{d} (\sigma_i^{j,0})^2 - (\sigma_i^{j,1})^2 \right) \right] F_t
$$

$$
= A + B \cdot F_t
$$

(14)

$$
\sigma_T^2(J) = \sum_{j=1}^{d} \left( F_i \cdot \left( \sum_{i=1}^{d} a_i \sigma_i^{j,0} \right)^2 + (t - F_i) \cdot \left( \sum_{i=1}^{d} a_i \sigma_i^{j,1} \right)^2 \right)
$$

$$
= \left( \sum_{j=1}^{d} \left( \sum_{i=1}^{d} a_i \sigma_i^{j,1} \right)^2 t \right) + \left[ \sum_{j=1}^{d} \left( \left( \sum_{i=1}^{d} a_i \sigma_i^{j,0} \right)^2 - \left( \sum_{i=1}^{d} a_i \sigma_i^{j,1} \right)^2 \right) \right] F_t
$$

$$
= C + E \cdot F_t
$$

(15)

Where $F_t = \int_{0}^{t} J_s ds$.

Remark: The distribution of $\Pi(t)$ (given $\Pi(0)$) does not fit into a known family (unless conditional on $J_t$), this is not a drawback for pricing purposes, expectations can be computed by simulating the jump process $J_t$. i.e. for CFO pricing, we can proceed as follow:

$$
P_0(\Pi(t) < D) = E[1_{\{\Pi(t) < D\}} | \Pi(0)]
$$

$$
= E \left[ E[1_{\{\Pi(t) < D\}} | J, W(0)] | J(0) \right]
$$

$$
= E \left[ \Phi \left( \frac{D - \mu_T(J)}{\sigma_T(J)} \right) | J(0) \right]
$$

$$
= E \left[ G(F_t) | J(0) \right]
$$

(16)

Where $G(F_t) = \Phi \left( \frac{D - \mu_T(J)}{\sigma_T(J)} \right) = \Phi \left( \frac{D - A + B \cdot F_t}{C + E \cdot F_t} \right)$, $A, B, C, E$ were defined in property 3. Then we simulate paths of $J$ (MonteCarlo) in order to compute $G(F_t)$.

**Estimation**

Hedge funds report returns on a monthly basis leading to a lack of data for estimation purposes. Therefore it becomes necessary to consider multivariate models with as few parameters as possible. One way to achieve this is by describing the dependence structure of the underlying Multivariate Brownian processes using factor analysis.
The parameters of the model are estimated using the following two step approach:

1 - Search for the times where a jump has occurred. Here we start with an initial guess for distress months described before, then a sample point is removed if a test rejected the gaussianity hypothesis. We stopped as soon as gaussianity is accepted. Knowing the distress and the tranquil months, allow us to compute the parameters of the jump process \((p, \lambda)\) by fitting the theoretical probability path of a tranquil months to the empirical probability path:

From equation [8], the theoretical probability of a tranquil month at time \(t\) given \(J_0 = 0\) is:

\[
q(t) = p \left(1 - e^{-\lambda t}\right)
\]

The empirical probability would be: \(\tilde{q}(t) = \sum_{i=1}^{t} \frac{NT(t)}{t}\), where \(NT(t)\) is the number of tranquil months by \(t\). Therefore fitting the theoretical to the empirical we get:

\[
p = 0.867 \\
\lambda = 1.029
\]

2 - Factor analysis is used for estimating each of the covariance matrices for tranquil and distress events. We use both sample data found in step 1 (returns from tranquil months and from distress month). Specifically, we propose a 1 factor model as follow:

\[
c^0 \cdot dW_Q(t) = \lambda^0 \cdot dM^Q(t) + B^0 \cdot dZ^Q(t) \\
c^1 \cdot dW_Q(t) = \lambda^1 \cdot dM^Q(t) + B^1 \cdot dZ^Q(t)
\]

where \(c^i\) is the correlation matrices under tranquil \((i = 0)\) and distress \((i = 1)\) conditions. \(\lambda^0, \lambda^1\) are constant vectors and \(B^i\) are diagonal matrices with diagonal \(\sqrt{1 - \left(\lambda_j(1)\right)^2}\). This reduces the total number of parameters to be computed for the covariance matrices from \(n^2 + n\) to \(4n\). Means are computed using standard estimators.
Exhibit 1: Schematic prototype of a CFO structure. Typical maturities 3-7 years.

Exhibit 2: Correlation switching phenomenon. Each pixel represents a correlation number between the funds, one from each axis. As we can see, it is apparent that in distressed periods the correlations between the
different managers tend to be greater in absolute value than in normal or tranquil periods.

Exhibit 3: Schematic representation of the CFO valuation model. Probabilities of default are given by a percentile of the pool of hedge funds returns distribution. Recovery rates distribution and spread calculations are easy to obtain within this model.

<table>
<thead>
<tr>
<th></th>
<th>R_1</th>
<th>R_2</th>
<th>R_3</th>
<th>R_4</th>
<th>R_5</th>
<th>R_6</th>
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<tbody>
<tr>
<td>( \mu )</td>
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<td>0.00520</td>
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<tr>
<td>( \sigma_1^2 )</td>
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<td>0.0039</td>
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<td>0.0078</td>
<td>0.0114</td>
<td>0.00313</td>
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<tr>
<td>( \sigma_2^2 )</td>
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<td>0.001</td>
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Exhibit 4: Mean and Variances.
Exhibit 5: Lambdas in Factor Analysis.

<table>
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<th></th>
<th>R₁</th>
<th>R₂</th>
<th>R₃</th>
<th>R₄</th>
<th>R₅</th>
<th>R₆</th>
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<tbody>
<tr>
<td>λ₀</td>
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<td>0.458</td>
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<tr>
<td>λ₁</td>
<td>0.862</td>
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<td>0.981</td>
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Exhibit 5: Tranquil Correlation.

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Exhibit 5: Distress Correlation.

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<th>R₅</th>
<th>R₆</th>
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</table>

Exhibit 6: Estimated probability of default and spread yield for the tranche.

<table>
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<th>3</th>
<th>2</th>
<th>1</th>
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</thead>
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<td>sₘ</td>
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<td>2.3</td>
<td>2.1</td>
<td>1.14</td>
<td>0.1</td>
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</tbody>
</table>
Figure 1: Exhibit 7: Sensitivities of probabilities of default to market conditions.

Figure 2: Exhibit 8: Spread yield sensitivities to market conditions.