A note on Laplace's equation inside a cylinder

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Abstract

Two difficulties connected with the solution of Laplace's equation around an object inside an infinite circular cylinder are resolved. One difficulty is the non-convergence of Fourier transforms used, in earlier publications, to obtain the general solution, and the second difficulty concerns the existence of apparently different expressions for the solution. By using a Green's function problem as an easily analyzed model problem, we show that, in general, Fourier transforms along the cylinder axis exist only in the sense of generalized functions, but when interpreted as such, they lead to correct solutions. We demonstrate the equivalence of the corrected solution to a different general solution, also previously published, but we point out that the two solutions have different numerical properties.

Keywords

Laplace equation; Fourier Transforms; Green's function; Sphere in Cylinder; Generalized functions;

1 Introduction

This paper addresses the methods used to study the electric field around a cavity in a wire, or the fluid motion around a drop inside a pipe, or similar problems in which an object is placed inside an infinite cylinder. Several papers have presented solutions for the field around a spherical cavity or drop in a cylinder, the most recent paper being Linton [1] and the earliest being Knight [2]. These works overlooked the fact that their integral transforms do not converge in all cases. This non-convergence can be demonstrated without solving the full problem of a sphere inside a cylinder, because the convergence problem is already present in the simplest problem that can be posed: the Green's function for Laplace's equation in a cylinder with Neumann boundary conditions, which in physical terms means the electric field created by a point charge, or the ideal flow from a point source. This paper uses the derivation of the Green's function as a model problem, which allows us to pinpoint the difficulty and its resolution, without the distractions of the more complicated full problem (of a finite-sized body in a cylinder).

Having shown the existence of a convergence problem in the Knight—Linton approach, we give a remedy. We must bear in mind, when considering possible remedies, that the Green's function problem used here is a model problem, and any method proposed must generalize back to the problems originally considered by Knight and Linton. We show that the Knight—Linton approach can be repaired using generalized functions, and then their method remains viable. We also point out an alternative method, outlined by Morse & Feshbach [3] for one simple set of problems. We show that the Morse—Feshbach method gives a solution equivalent to the Fourier transform method, but that the numerical properties of the two solutions are different. Morse—Feshbach has better properties for large z and Knight—Linton for small z. It should be realized, however, that the Morse—Feshbach method has not been tried on the spherical cavity problem, but only on the model problem given here.

The problem for the Green's function is as follows. We scale cylindrical coordinates (r, θ, z) so that the boundary conditions are imposed on $r = 1$. The Green's function satisfies

$$\nabla^2 G = -4\pi\delta(\mathbf{x}) \tag{1}$$

and the Neumann boundary condition $\partial G/\partial r = 0$ on $r = 1$. This is our model problem, and we wish to solve it in a way that illuminates the Knight—Linton approach. Jumping ahead to the solution, given below in equation (11), we shall see that asymptotically $G \sim -2|z|$ for large z owing to the Neumann boundary condition. In section 2 we consider the consequences of this.

2 Fourier transform method

Knight [2] and others effectively take a Fourier transform of (1) with respect to z. Since we have already stated that $G \sim -2|z| + o(1)$ for $z \to \infty$, a Fourier transform does not exist in the ordinary sense. In looking for a response to this difficulty, we must not be misled by the simplicity of the problem (1). It is tempting to consider deriving equations for $G + 2|z|$, a quantity whose transform would exist. However, for the more difficult problems considered by Knight [2] and Linton [1], the asymptotic behaviour of the solution is one of the main goals of the calculation. Therefore, although reformulating the problem in terms of convergent integrals would be a possibility in this model problem, it is a solution that does not generalize to harder problems. We can, however, continue to use Knight's method, provided we are later willing to interpret the integrals as generalized functions.

It is convenient to separate the singularity in G by writing

$$G = (r^2 + z^2)^{-1/2} + \varphi\ , \tag{2}$$

and considering the problem for φ, which is

$$\nabla^2\varphi = 0\ , \tag{3}$$

$$\frac{\partial\varphi}{\partial r} = (1 + z^2)^{-3/2} \quad \text{on} \quad r = 1\ . \tag{4}$$

As with G, the asymptotic behaviour of φ will be $-2|z|$ as $z \to \infty$. We note from (3), (4) that this problem is obviously symmetric in z; however, we do not take advantage of this symmetry to reformulate the problem for two reasons. First, the papers we are commenting on did not do it, and second, we wish to consider a method that would apply to non-symmetric situations. Also the difficulties we address are present even if one restricts the problem to $z \geq 0$.

If $\bar{\varphi}$ is the Fourier transform of φ with respect to z, it satisfies a modified Bessel equation

$$\frac{\partial^2 \bar{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\varphi}}{\partial r} - t^2 \bar{\varphi} = 0 ,$$

whose independent solutions are $I_0(tr)$ and $K_0(tr)$. Since K_0 is singular at $r = 0$, it is rejected, and the solution, symmetric with respect to z is

$$\varphi(r,z) = \int_0^\infty g(t) I_0(rt) \cos zt \, dt , \tag{5}$$

with $g(t)$ to be determined from the boundary condition. By differentiating (see [4])

$$\int_0^\infty K_0(rt) \cos zt \, dt = \frac{\pi}{2}(r^2 + z^2)^{-1/2} , \tag{6}$$

we deduce that (4) is apparently satisfied by setting $g(t) = (2/\pi) K_1(t)/I_1(t)$. We combine (6) and (2) to write the Green's function finally as

$$G(r,z) = \frac{2}{\pi} \int_0^\infty \left(K_0(rt) + \frac{K_1(t)}{I_1(t)} I_0(rt) \right) \cos zt \, dt . \tag{7}$$

The problem, now, is to rewrite (7) as a convergent integral, because for small t, the integrand has the expansion

$$\frac{K_1(t)}{I_1(t)} I_0(rt) \to \frac{2}{t^2} + O(1) , \quad \text{for } t \to 0 , \tag{8}$$

and therefore the integral does not converge. However, the theory of generalized functions allows us to write [5]

$$\int_0^\infty \frac{1}{t^2} \cos tz \, dt = -\frac{\pi}{2}|z| . \tag{9}$$

This result could also be obtained using the concept of Hadamard's finite part (see [6]). The connexion between the theory of generalized functions (distributions) and Hadamard's finite part is described in [5, 6]. Hence (7) can be rewritten as

$$G(r,z) = -2|z| + \frac{2}{\pi} \int_0^\infty \left(K_0(rt) + \frac{K_1(t)}{I_1(t)} I_0(rt) - \frac{2}{t^2} \right) \cos zt \, dt . \tag{10}$$

Thus the approach of Knight and Linton can be used with this re-interpretation.

3 Morse & Feshbach's solution

A different solution for the Green's function (2) is given in Morse & Feshbach [3]. They describe the problem as flow in $z > 0$ when fluid enters the cylinder from a small hole in a wall at $z = 0$. Using separation of variables, they obtain a solution in terms of Bessel functions $J_0(\beta_k r)$, with β_k defined by $J_1(\beta_k) = 0$. In present notation, we normalize their problem by setting flux/unit area $= 1$ and obtain

$$G(r,z) = -2z + \sum_{k=1}^{\infty} \frac{2}{\beta_k J_0(\beta_k)^2} e^{-\beta_k z} J_0(\beta_k r) \ . \tag{11}$$

Before showing the equivalence of (11) and (7), we comment that both solutions can be understood in terms of the technique of separation of variables. When separating Laplace's equation in cylindrical coordinates, one can take the constant of separation as positive, in which case we are led to (11), or negative, in which case we obtain (7). Introductory courses on partial differential equations typically explore only one choice for the constant of separation.

We explicitly convert solution (7) to (11) by expanding the integrand in (10) as a Dini-Bessel series, valid for $0 < r < 1$,

$$K_0(rt) + \frac{K_1(t)}{I_1(t)} I_0(rt) - \frac{2}{t^2} = \sum_{k=1}^{\infty} \frac{2}{J_0^2(\beta_k)} \frac{1}{t^2 + \beta_k^2} J_0(\beta_k r) \ ,$$

where the coefficients were derived using the formula [4]

$$\int_0^1 r K_n(tr) J_n(\lambda r) dr = \frac{1}{t^2 + \lambda^2} \left[\left(\frac{\lambda}{t} \right)^n + \lambda J_{n+1}(\lambda) K_n(t) - t J_n(\lambda) K_{n+1}(t) \right] \ ,$$

which is valid for $n > -1$. Using this expansion in (10), we can integrate term by term, using the formula [4]

$$\int_0^{\infty} \frac{\cos tz}{t^2 + \beta_k^2} \, dt = \frac{\pi}{2\beta_k} e^{-\beta_k |z|} \ ,$$

and conclude that the two expressions are equivalent.

4 Numerical properties of the solutions

Although the two expressions are equivalent, they have different (numerical) convergence properties: (7) converges slowly for large z, while (11) converges slowly for small z. In (11), the exponential terms $e^{-\beta_k |z|}$ will all tend to 1 as $z \to 0$, and the expansion will be slowly convergent when z is small. On the other hand, the $\cos(zt)$ factor in (7) will cause numerical difficulties for large z, because it will oscillate rapidly.

5 Dipole on axis

In many applications, the dominant response of a sphere or a similar object in a cylinder will be as a dipole rather than as a pole [1]. We note here that convergence problems, similar to those observed in the case of the pole, persist in the dipole case. By differentiating (7) with respect to z, we obtain the potential ϕ of a unit dipole in a form equivalent to Linton's

$$\phi(r,z) = -\frac{2}{\pi}\int_0^\infty t\left(K_0(rt) + \frac{K_1(t)}{I_1(t)}I_0(rt)\right)\sin zt\;dt\;. \tag{12}$$

The integrand is asymptotically $t(K_0(rt)+K_1(t)/I_1(t)I_0(rt)) \sim 2/t$ and the integral again converges only in the sense of generalized functions. Either by separating the singular behaviour in (12) or by differentiating (11), we see that

$$\phi(r,z) \sim -2\,\mathrm{sgn}\,z + o(1) \text{ as } z \to \infty\;.$$

6 Conclusions

We have considered two general methods for solving Laplace's equation around an object in a cylinder. We have shown that a re-interpretation of the transforms used by Knight and Linton allows their method to be used reliably. Further, we showed that the different methods offer different numerical properties, although the Morse—Feshbach method has only been applied to the problems described here, and not yet to more complicated situations.

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