A NOTE ON THE ONE-COLOUR AVOIDANCE GAME ON GRAPHS

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Abstract. We consider the one-colour graph avoidance game. Using a high performance computing network, we showed that the first player can win the game on 13, 14, and 15 vertices. Other related games are also discussed.

1. Introduction

In this note, we consider the one-colour graph avoidance game. Let G be a fixed graph on n_0 vertices, and let $n \geq n_0$ be an integer. The game between two players, first and second, starts with n isolated vertices. In each turn, a player draws a new edge. Players alternate turns, starting with the first player. We deal with simple graphs only so it is forbidden to create parallel edges and loops. Both players have the same goal, namely to try to avoid creating G as a subgraph. Since $n \geq n_0 = |V(G)|$, it is unavoidable and eventually one player is forced to create a copy of G and lose the game. Since this is a two-person, full information game with no ties, either the first player or the second player has a winning strategy (namely, the strategy to achieve a maximal G-free graph, that is, a G-free graph with the property that the addition of any edge creates a copy of G).

We consider also two natural variants of the game we described. In the first one, the players try to avoid all graphs from a given family $\mathcal{G} = \{G_i : i \in S\}$ (S can be finite or infinite). In the second one, every edge played after the first move must be adjacent to some previously played edge. In other words, at each stage of the game after the first move all components but one are isolated vertices. This is known as the *connected variant* of the graph avoidance game.

2. Avoiding a triangle

The well-known, interesting, and highly nontrivial game is when the players try to avoid creating a triangle (see [3] for more details on this and other variations). The outcomes for $n \leq 9$ were reported by Seress [6]. For several years only two more values were known (namely, for $n = 10$ and $n = 11$) and it was conjectured that the first player wins if, and only if, $n \equiv 2$ (mod 4). However, Cater, Harary, and Robinson [2] managed to show, using computer support, that the conjecture fails for $n = 12$. It seems that there is no hope to solve the problem completely and to show who has a winning strategy for any n. On the other hand, it has been shown that the first player wins the connected variant of the game if, and only if, n is even [6].

In this paper we report that the first player wins the game on 13, 14, and 15 vertices and conjecture that this is the case for each $n \geq 12$. (We do admit that the conjecture

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is rather bold; the only reason to do so, beyond the values in Table 1, is that almost all combinatorial games are first player win [7].) In order to obtain this result, we generated a family $\mathcal{H}$ of all nonisomorphic triangle-free graphs on n vertices using Brendan McKay's `nauty` software package [5] for computing automorphism groups of graphs and digraphs. Let $h(n) = |\{G : G \in \mathcal{H}\}|$ denote the number of such graphs and let $e(n) = \max\{|E(G)| : G \in \mathcal{H}\}$ be the number of edges in the densest graph in this family. Let us note that $e(n) = \lfloor n^2/4 \rfloor$ by Mantel's theorem (an upper bound follows from Turán's theorem; for a lower bound consider the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$). It is clear that all graphs with $e(n)$ edges have the property that the next player to move loses the game. We call those graphs *previous player wins* graphs, and denote corresponding subfamily by $P_{e(n)}$. Now, we partition the set of all graphs on $e(n) - 1$ edges into *previous player wins* graphs ($P_{e(n)-1}$) and *next players wins* ones ($N_{e(n)-1}$). In order for a graph G to be in $N_{e(n)-1}$, it is required that there is an edge $e \notin E(G)$ such that after adding e to G we get a graph which is in $P_{e(n)}$. (The next player should draw e to force the opponent to give up.) Since this is also a sufficient condition,

$$\begin{aligned} N_{e(n)-1} &= \{G \in \mathcal{H} : G = H \setminus \{e\} \text{ for some } H \in P_{e(n)} \text{ and } e \in E(H)\} \\ P_{e(n)-1} &= \{G \in \mathcal{H} : |E(G)| = e(n) - 1\} \setminus N_{e(n)-1}. \end{aligned}$$

Those operations can be done easily with the support of the `nauty` software package to remove isomorphisms. Now, we can determine the families P_i and N_i ($i = e(n) - 2, e(n) - 3, \ldots, 0$) recursively. If the only graph with no edge in $\mathcal{H}$ (empty graph) is in N_0, then the first player wins the game (we put $w(n) = 1$); otherwise the second player has a winning strategy ($w(n) = 2$). A UNIX script used to solve the problem can be found in [8]. Below we present the result of our program (Table 1).

n	$w(n)$	$e(n)$	$h(n)$
3	2	2	3
4	2	4	7
5	2	6	14
6	1	9	38
7	2	12	107
8	2	16	410
9	2	20	1,897
10	1	25	12,172
11	2	30	105,071
12	1	36	1,262,180
13	1	42	20,797,002
14	1	49	467,871,369
15	1	56	14,232,552,452

TABLE 1. Triangle avoidance game.

Let us finish this section with one more problem related to this game. Suppose that the player L who loses the game is trying to avoid a triangle for as long as possible; the goal of the other player, W, is the opposite. The payoff to L is the number of moves until triangle is created, and he obviously seeks the highest possible payoff. Let $f(n)$ denote the smallest payoff over all possible winning strategies of W, assuming L uses an optimal strategy.

Clearly, $f(n) \le e(n) = \lfloor n^2/4 \rfloor$. It follows from Tables 5–7 (see the Appendix) that $f(13) \le 37 < 42 = e(13)$, since all maximal triangle free graphs with an odd number of edges have at most 37 edges (recall that the first player has a winning strategy), and that $f(15) \le 49 < 56 = e(15)$. Unfortunately, no non-trivial upper bound for $f(14)$ can be obtained without investigating the game in a more sophisticated way. Nevertheless, it seems natural to conjecture that $\lim_{n\to\infty} \frac{f(n)}{e(n)} = 0$.

Consider the following stochastic process. We begin with the empty graph on n vertices. At each step of the process, we add an edge chosen uniformly at random from the collection of pairs of vertices that neither appears as existing edges nor form a triangle when added as edges. We know that asymptotically almost surely the process ends after $\Theta(n^{3/2}\sqrt{\log n})$ steps (that is, a maximal triangle free graph is obtained) [1]. Is it true that $f(n) = \Theta(n^{3/2}\sqrt{\log n})$?

3. Other related games

Using exactly the same approach as before and the same UNIX script, one can analyze the square (C_4) avoidance game and the game where players try to avoid cycles of length at most 4 (that is, $\mathcal{G} = \{C_3, C_4\}$). Both classic and connected variants are studied and the results are presented in Tables 2 and 3. In order to analyze a connected variant of the game, all "disconnected" graphs (that is, graphs with at least two nontrivial components) have to be removed at each stage of the process.

n	$w(n)$	$e(n)$	$h(n)$
4	2	4	8
5	1	6	18
6	1	7	44
7	1	9	117
8	1	11	351
9	1	13	1,230
10	1	16	5,069
11	1	18	25,181
12	1	21	152,045
13	2	24	1,116,403
14	2	27	9,899,865
15	1	30	104,980,369
16	1	33	1,318,017,549

(a) Classic version

n	$w(n)$	$h(n)$
4	2	7
5	1	15
6	1	34
7	2	91
8	1	277
9	1	1,017
10	1	4,406
11	1	22,908
12	1	143,129
13	2	1,075,389
14	2	9,672,233
15	2	103,434,937
16	1	1,305,167,374

(b) Connected variant

Table 2. C_4 avoidance game.

n	$w(n)$	$e(n)$	$h(n)$
4	1	3	6
5	2	5	11
6	2	6	23
7	2	8	48
8	1	10	114
9	2	12	293
10	1	15	869
11	1	16	2,963
12	1	18	12,066
13	2	21	58,933
14	2	23	347,498
15	2	26	2,455,693
16	2	28	20,592,932
17	2	31	202,724,920

(a) Classic version

n	$w(n)$	$h(n)$
4	1	5
5	2	9
6	1	17
7	2	35
8	1	82
9	2	219
10	1	683
11	1	2,476
12	1	10,643
13	1	54,288
14	2	329,768
15	1	2,375,047
16	2	20,147,694
17	1	199,741,517

(b) Connected variant

TABLE 3. C_3 and C_4 avoidance game.

Another natural related game is that of odd cycle avoidance, in which the first player to create an odd cycle loses the game (that is, $\mathcal{G} = \{C_{2k+1} : k \in \mathbb{Z}_+\}$). It has been shown that the first player wins the classic game if, and only if, $n \equiv 2 \ (mod\ 4)$ [2]. It is also not difficult to see that the following theorem holds.

Theorem 3.1. *The first player wins the connected variant of the odd cycle avoidance game on $n \geq 3$ if, and only if, n is even.*

Proof. It is clear that at each point of the game the graph we play on is bipartite. The game ends when a complete bipartite graph K_{n_1,n_2} ($n_1, n_2 \geq 1$ and $n_1 + n_2 = n$) is created so that the next player to move has to create an odd cycle and lose. If n is odd, one of n_1, n_2 is even and the number of edges in K_{n_1,n_2} (and the number of moves in the game at the same time) is exactly $n_1 n_2$ which is even. This implies that the second player wins the game if n is odd.

In order to show that the first player wins the game if n is even, we propose the following simple strategy. The first player can ensure that after each of his moves, the only nontrivial component is *safe*, that is, the two partite classes have odd sizes. This will finish the proof; $n_1 n_2$ is odd and after creating K_{n_1,n_2} the second player is forced to give up.

The claim is proved by induction. After the first move, the main component consists of an isolated edge and the base case holds. For the inductive step, assume that the main component is *safe*. Suppose first that the second player joins an isolated vertex x to vertex y from the main component so that it is not *safe* anymore. Since the number of vertices is even, there is at least one more isolated vertex z that can be joined to y by the first player to keep the component *safe*. Suppose now that the second player joins two vertices in the main component. We will show that the first player can also join two vertices in the main component so it remains *safe*. Indeed, suppose for a contradiction that he is forced to join an isolated vertex. This means that the main component at that point of the game is a complete bipartite graph. But, since the component is *safe*, the number of edges in the complete bipartite graph is odd which gives a contradiction (the number of moves/edges up to this point is even). □

Finally, let us present results on the game where players try to avoid odd cycles and the square, that is, $\mathcal{G} = \{C_{2k+1} : k \in \mathbb{Z}_+\} \cup \{C_4\}$ (Table 4). This game is motivated by the Zarankiewicz problem which asks how many edges can be added to a bipartite graph while avoiding a specific bipartite subgraph. The Kővári-Sós-Turán theorem gives a bound on the Zarankiewicz problem when the subgraph to be avoided is a complete bipartite graph [4]. For example, a bipartite graph with $2n$ vertices and no 4-cycles has $O(n^{3/2})$ edges. This bound is within a constant factor of optimal, as there exists a bipartite graph based on the projective plane that has $\Omega(n^{3/2})$ edges. (For a fixed prime power q, let $G_q = (P, L, E)$ be a bipartite graph with bipartition P, L where P and L denote the set of points and, respectively, lines in the projective plane. A point is joined to a line if it is contained in it. Then G_q has $2(q^2 + q + 1)$ many vertices and is $(q + 1)$-regular.)

Introducing functions $f(n), e(n)$ for this game as before, we get that $f(n) \leq e(n) = \Theta(n^{3/2})$. One could try to prove that $\lim_{n\to\infty} \frac{f(n)}{e(n)} = 0$. Moreover, based on the values in Table 4(b), one could conjecture that $w(n) = 1$ if, and only if, n is even.

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The programs used to obtain the result can be downloaded from [8]. The total computational requirements to solve the triangle avoidance game for $n = 15$ we estimated to be 1,440 CPU hours (2 months). Other results are much easier to obtain.

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n	$w(n)$	$e(n)$	$h(n)$
4	1	3	6
5	2	4	10
6	1	6	21
7	2	7	39
8	1	9	86
9	2	10	182
10	1	12	440
11	2	14	1,074
12	1	16	2,941
13	2	18	8,424
14	1	21	26,720
15	2	22	90,883
16	1	24	340,253
17	2	26	1,384,567
18	2	29	6,186,907
19	2	31	30,219,769
20	1	34	161,763,233
21	2	36	946,742,190

(a) Classic version

n	$w(n)$	$h(n)$
4	1	5
5	2	8
6	1	15
7	2	27
8	1	57
9	2	121
10	1	298
11	2	755
12	1	2,158
13	2	6,485
14	1	21,509
15	2	76,239
16	1	295,550
17	2	1,238,458
18	1	5,664,971
19	2	28,189,986
20	1	153,086,780
21	2	906,020,372

(b) Connected variant

Table 4. C_4 and odd cycle avoidance game.

5. Appendix

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i	$\|P_i\|$	$\|N_i\|$	i	$\|P_i\|$	$\|N_i\|$
0	0	1	22	589,428	2,440,325
1	1	0	23	728,346	1,733,458
2	0	2	24	408,653	1,313,910
3	4	0	25	344,083	705,468
4	0	9	26	176,223	391,913
5	18	1	27	92,534	186,717
6	1	44	28	55,414	72,767
7	97	7	29	16,228	39,687
8	8	254	30	14,034	9,741
9	557	118	31	2,126	7,704
10	63	1,745	32	3,016	1,063
11	2,941	1,958	33	237	1,402
12	723	12,556	34	576	99
13	14,071	20,957	35	24	245
14	8,076	80,425	36	109	9
15	60,729	147,866	37	3	40
16	57,078	392,781	38	19	0
17	218,032	652,881	39	0	7
18	223,016	1,267,998	40	4	0
19	553,047	1,674,464	41	0	1
20	485,087	2,392,364	42	1	0
21	846,195	2,345,213		4,900,802	15,896,200

TABLE 5. Triangle avoidance game on 13 vertices – more details.

i	$\lvert P_i \rvert$	$\lvert N_i \rvert$	i	$\lvert P_i \rvert$	$\lvert N_i \rvert$
0	0	1	26	7,363,526	46,025,778
1	1	0	27	15,012,885	23,391,750
2	0	2	28	4,052,865	20,318,024
3	4	0	29	6,242,870	7,593,500
4	0	9	30	1,295,418	5,860,711
5	19	0	31	1,823,812	1,616,366
6	0	45	32	303,600	1,263,284
7	105	0	33	413,237	273,886
8	0	265	34	64,601	228,666
9	689	2	35	78,530	44,171
10	1	1,892	36	13,248	37,350
11	5,331	24	37	13,358	7,275
12	8	15,588	38	2,566	5,753
13	45,407	226	39	2,135	1,207
14	47	132,234	40	474	873
15	367,112	4,712	41	342	201
16	689	992,324	42	87	136
17	2,286,986	185,146	43	58	34
18	22,079	5,622,043	44	16	21
19	8,471,611	3,178,199	45	11	6
20	437,448	21,048,208	46	3	4
21	17,918,821	17,148,707	47	2	1
22	2,818,704	47,451,336	48	1	1
23	24,909,644	38,026,461	49	1	0
24	6,780,648	61,775,022			
25	23,759,354	41,111,571		124,508,354	343,363,015

TABLE 6. Triangle avoidance game on 14 vertices – more details.

i	$\lvert P_i\rvert$	$\lvert N_i\rvert$	i	$\lvert P_i\rvert$	$\lvert N_i\rvert$
0	0	1	29	410,418,030	1,210,382,897
1	1	0	30	245,748,052	972,418,343
2	0	2	31	235,296,884	580,809,896
3	4	0	32	119,812,722	372,666,826
4	0	9	33	85,407,004	185,945,250
5	19	0	34	43,769,831	94,943,301
6	0	45	35	20,883,865	45,973,590
7	102	3	36	13,306,206	17,528,616
8	2	264	37	3,740,899	10,016,187
9	610	87	38	3,449,149	2,537,209
10	47	1,881	39	528,324	2,020,933
11	3,679	1,874	40	760,394	306,059
12	1,283	15,441	41	62,712	375,043
13	23,487	28,256	42	144,912	32,409
14	21,097	141,735	43	6,640	63,906
15	156,900	355,059	44	24,812	3,142
16	274,669	1,305,764	45	661	10,217
17	1,050,178	3,649,591	46	4,012	283
18	2,513,178	10,723,860	47	64	1,595
19	6,507,950	28,265,080	48	650	24
20	15,621,442	68,482,373	49	7	254
21	33,175,092	152,068,065	50	110	0
22	64,466,478	303,754,074	51	0	40
23	123,515,139	532,151,068	52	19	0
24	171,538,611	868,077,711	53	0	7
25	297,014,462	1,163,873,140	54	4	0
26	297,831,236	1,515,119,857	55	0	1
27	443,313,245	1,539,153,335	56	1	0
28	337,166,250	1,571,786,724		2,977,561,125	11,254,991,327

Table 7. Triangle avoidance game on 15 vertices – more details.