ON AN ADJACENCY PROPERTY OF ALMOST ALL TOURNAMENTS

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In loving memory of Claude Berge.

ABSTRACT. Let n be a positive integer. A tournament is called n-existentially closed (or n-e.c.) if for every subset S of n vertices and for every subset T of S, there is a vertex $x \notin S$ which is directed toward every vertex in T and directed away from every vertex in $S \backslash T$. We prove that there is a 2-e.c. tournament with k vertices if and only if $k \geq 7$ and $k \neq 8$, and give explicit examples for all such orders k. We also give a replication operation which preserves the 2-e.c. property.

1. Introduction

A *tournament* is a directed graph with exactly one arc between each pair of distinct vertices. Consider the following *adjacency property* for tournaments.

Definition 1. *Let n be a positive integer. A tournament is called n-**existentially closed** or n-**e.c.** if for every n-element subset S of the vertices, and for every subset T of S, there is a vertex $x \notin S$ which is directed toward every vertex in T and directed away from every vertex in $S \backslash T$. (Note that T may be empty.)*

Adjacency properties of tournaments were studied in [3, 7, 13, 16, 21]. Much of the research on such properties is motivated by the fact that while almost all tournaments (with arcs chosen independently and with probability p, where $0 < p < 1$ is a fixed real number) are n-e.c. for any fixed positive integer n (see [13]), few *explicit* examples of such tournaments are known.

Adjacency properties of graphs were studied by numerous authors; see [8] for a survey. A graph is called *n-existentially closed* or *n-e.c.* if

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it satisfies the following adjacency property: for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex of T and to no vertex of $S \setminus T$. The n-e.c. property is of interest in part because the countable random graph is n-e.c. for all $n \geq 1$; in fact, the countable random graph is the unique (up to isomorphism) countable graph that is n-e.c. for all $n \geq 1$. The countable random tournament is the analogue of the random graph for tournaments; see [11]. The countable random tournament is the unique (up to isomorphism) countable tournament that is n-e.c. for all $n \geq 1$.

The cases $n = 1, 2$ for graphs were studied in [8, 9, 10]. For $n > 2$, few explicit examples of n-e.c. graphs are known other than large Paley graphs (see [2, 7]). A prolific construction of n-e.c. graphs for all n was recently given in [12].

In the present article, we concentrate on the 2-e.c. adjacency property. Note that a tournament is 2-e.c. if the following adjacencies hold: for every pair of vertices, u and v, there are four other vertices: one directed toward both u and v, one directed away from both u and v, one directed toward u and away from v, and one directed toward v and away from u. In Section 3, we prove that there is a 2-e.c. tournament with k vertices if and only if $k \geq 7$ and $k \neq 8$, and give explicit examples for all such orders k.

We consider only finite and simple tournaments. For a tournament G, $V(G)$ denotes its vertex-set and $E(G)$ denotes its arc-set. The order of G is $|V(G)|$. We denote an arc directed from x to y by (x, y). For a vertex $x \in V(G)$, we define $N_{out}(x) = \{y : (x, y) \in E(G)\}$, and $N_{in}(x) = \{y : (y, x) \in E(G)\}$. As usual, a vertex x with $N_{in}(x) = \emptyset$ is called a source and a vertex x with $N_{out}(x) = \emptyset$ is called a sink. If $U \subseteq V(G)$, $G \upharpoonright U$ is the subgraph of G induced by U; for $x \in V(G)$, $G - x = G \upharpoonright (V(G) \setminus \{x\})$.

The *Paley tournament* of order q, written D_q, where q is a prime power congruent to 3 (mod 4), is the tournament with vertices the elements of $GF(q)$, the finite field with q elements, and $(x, y) \in E(D_q)$ if and only if $x - y$ is a nonzero quadratic residue. For D_7, see Figure 1. As discussed above for Paley graphs, for a fixed positive n, sufficiently large Paley tournaments are n-e.c. (see [16]); however, no other explicit families of tournaments with these adjacency properties are known.

The next lemma follows from the definitions.

Lemma 1. *Let G be an n-e.c. tournament for some $n > 1$. For a fixed $v \in V(G)$, the tournaments $G - v$, $G \upharpoonright N_{in}(v)$, and $G \upharpoonright N_{out}(v)$ are each $(n-1)$-e.c.*

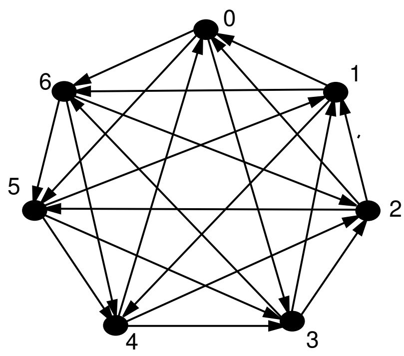

FIGURE 1. An isomorph of D_7.

Definition 2. *A tournament G is n-**e.c. minimal** if G has the smallest number of vertices among all n-e.c. tournaments. An n-e.c. tournament is **critical** if deleting any vertex leaves a tournament which is not n-e.c.*

Clearly, an n-e.c. minimal tournament is n-e.c. critical. In Section 2, we show that there are exactly two 1-e.c. critical tournaments up to isomorphism. In Section 4, we give examples of 2-e.c. critical tournaments of all possible orders $k \geq 7$ and $k \neq 8$. Vertex-criticality for various properties has been studied by many authors, including Berge [4, 5, 6] and [1, 15, 18, 19, 20, 22, 23].

2. THE 1-E.C. CRITICAL TOURNAMENTS

We make the following trivial observations.

Remark 1. *A tournament is 1-e.c. if and only if it has no source or sink.*

Remark 2. *A tournament with a directed hamilton cycle is 1-e.c.*

The tournament D_3 is the directed circuit on three vertices. It is easy to see that D_3 is the unique (up to isomorphism) 1-e.c. minimal tournament, and thus, it is 1-e.c. critical. Define T_6 to be the tournament consisting of two copies of D_3, with arcs oriented from the first copy to the second. It is straightforward to check that T_6 is 1-e.c. critical.

Theorem 2. *The only 1-e.c. critical tournaments (up to isomorphism) are D_3 and T_6.*

Proof. Let G be a 1-e.c. critical tournament. We first observe that a strongly connected component S of G has exactly 3 vertices. To see this, suppose that S has at least $k \geq 4$ vertices. By a theorem of Moon [17], S has a directed circuit C of length $k-1$. Deleting the vertex that C misses in S leaves a 1-e.c. tournament, which is a contradiction.

We claim that if G has exactly one or two strongly connected components, then G is isomorphic to D_3 or T_6, respectively. Assume to the contrary that G has $r \geq 3$ strongly connected components. From G we construct an auxiliary tournament G', whose vertices are the strongly connected components of G with the induced adjacencies. Note that G' is isomorphic to the r-element linear order. Let u be a vertex of G' that is neither a least nor greatest element. If we delete a vertex x in the strongly connected component of G corresponding to u, then the remaining graph $G - x$, is 1-e.c., which is a contradiction. □

3. Examples of 2-e.c. tournaments

In this section, our main theorem is the following.

Theorem 3. *There is a 2-e.c. tournament with k vertices if and only if $k \geq 7$ and $k \neq 8$.*

To prove Theorem 3, we first prove the following theorem.

Theorem 4. *There is a unique (up to isomorphism) 2-e.c. minimal tournament, the Paley tournament D_7.*

Proof. Let G be a 2-e.c. tournament. Then since the unique minimal 1-e.c. tournament has 3 vertices, $|V(G)| \geq 7$ by Lemma 1. Suppose now $|V(G)| = 7$, say $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$. Say $N_{in}(7) = \{1, 2, 3\}$, $(1, 2), (2, 3), (3, 1) \in E(G)$; $N_{out}(7) = \{4, 5, 6\}$; $(4, 6), (6, 5), (5, 4) \in E(G)$. See Figure 2 (a). Vertex 1 currently has outdegree two, but needs outdegree three, so without loss of generality, assume that $(1, 4) \in E(G)$. Then by considering the degrees of 1 and 4, we get $(5, 1), (6, 1) \in E(G)$ and $(4, 2), (4, 3) \in E(G)$. See Figure 2 (b). Since $N_{in}(1) = \{3, 5, 6\}$ and $(6, 5) \in E(G)$, it follows that $(5, 3), (3, 6) \in E(G)$. See Figure 2 (c). Then, for degree of 5, $(2, 5) \in E(G)$, and then for degree of 2, $(6, 2) \in E(G)$. See Figure 2 (d). Then $f : V(G) \to V(D_7)$ is an isomorphism, where $f(1) = 0, f(2) = 5, f(3) = 4, f(4) = 6, f(5) = 1, f(6) = 2$ and $f(7) = 3$. □

Given a 2-e.c. tournament, another 2-e.c. tournament with two more vertices can be constructed using a "tournament version" of the replication operation which was instrumental in [8].

Definition 3. *Let G be a tournament and let $(a, b) \in E(G)$. Add two new vertices a', b' such that a' has the same adjacencies to vertices of G other than b as a does, b' has the same adjacencies to vertices of*

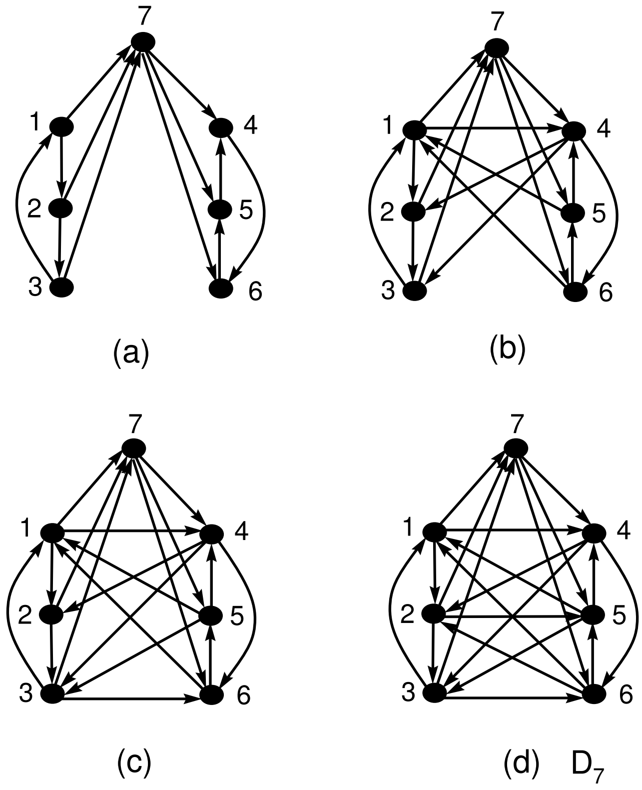

FIGURE 2. The proof of Theorem 4.

G other than a as b does, a, b, a', b', a is a directed circuit, a and a' are joined either way and b and b' are joined either way; that is, a ***replicate*** *$R = R(G, e)$ is a tournament with $V(R) = V(G) \cup \{a', b'\}$ and*

$$\begin{aligned} E(R) &= E(G) \cup \{(a', v) : v \in N_{out}(a) \setminus \{b\}\} \cup \{(v, a') : v \in N_{in}(a)\} \\ &\cup \{(b', v) : v \in N_{out}(b)\} \cup \{(v, b') : v \in N_{in}(b) \setminus \{a\}\} \\ &\cup \{(b, a'), (a', b'), (b', a)\} \cup \{\textit{exactly one of } (a, a'), (a', a)\} \\ &\cup \{\textit{exactly one of } (b, b'), (b', b)\}. \end{aligned}$$

We observe that for each arc e, there are four nonidentical replicates $R(G, e)$ that we may construct (depending on how we orient the edges aa', bb').

Definition 4. *Let G be a tournament, and let $n \geq 1$ be fixed.*

(1) *An n-**e.c. tournament problem** is a $2 \times n$ matrix*

$$\begin{pmatrix} x_1 & \dots & x_n \\ i_1 & \dots & i_n \end{pmatrix}$$

where $\{x_1, \dots, x_n\}$ is an n-element subset of $V(G)$, and for $1 \leq j \leq n, i_j \in \{\uparrow, \downarrow\}$.

(2) *A **solution** to an n-e.c. tournament problem is a vertex $z \in V(G)$ such that $z \in N_{in}(x_j)$ if $i_j = \uparrow$ and $z \in N_{out}(x_j)$ if $i_j = \downarrow$.*

Note that a tournament G is n-e.c. if and only if each n-e.c. tournament problem in G has a solution.

Theorem 5. *If G is a 2-e.c. tournament, then for every $e \in E(G)$, each replicate $R = R(G, e)$ is 2-e.c.*

Proof. Fix $e = (a, b) \in E(G)$. Fix distinct $x, y \in V(R)$. We show that each problem $\begin{pmatrix} x & y \\ i & j \end{pmatrix}$, $i, j \in \{\uparrow, \downarrow\}$ has a solution in R.

Case 1. $|\{a', b'\} \cap \{x, y\}| = 0$. A solution to the problem in G is a solution to the problem in R.

Case 2. $|\{a', b'\} \cap \{x, y\}| = 1$.

Assume that $x = a'$ and $y \neq b'$. First suppose $y = a$. If $(i, j) = (\uparrow, \uparrow)$, an in-neighbour of a in G solves the problem; if $(i, j) = (\downarrow, \downarrow)$, an out-neighbour of a in G other than b solves the problem. The vertex b solves $\begin{pmatrix} a' & a \\ \uparrow & \downarrow \end{pmatrix}$ and b' solves $\begin{pmatrix} a' & a \\ \downarrow & \uparrow \end{pmatrix}$.

If $y \neq a$, first solve $\begin{pmatrix} a & y \\ i & j \end{pmatrix}$ by say, c, in G. If $c \neq b$, then c also solves $\begin{pmatrix} a' & y \\ i & j \end{pmatrix}$. If $c = b$, then $i = \downarrow$ and $y \neq b$, so b' solves $\begin{pmatrix} a' & y \\ \downarrow & j \end{pmatrix}$.

The case when $x = b'$ and $y \neq a'$ follows by a similar argument.

Case 3. $|\{a', b'\} \cap \{x, y\}| = 2$.

Where z is a solution of $\begin{pmatrix} a & b \\ i & j \end{pmatrix}$ in G, z is a solution of $\begin{pmatrix} a' & b' \\ i & j \end{pmatrix}$ in R. □

Using tournament replication on D_7, we obtain 2-e.c. tournaments for any odd order k, $k \geq 7$. Now we work on finding 2-e.c. tournaments of all possible even orders.

Theorem 6. *There is no 2-e.c. tournament of order 8.*

Proof. It is straightforward to see that there is a unique 1-e.c. tournament of order 4; see Figure 3. Let G be a 2-e.c. tournament of order 8.

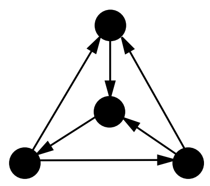

FIGURE 3. The unique 1-e.c. tournament of order 4.

Then G has a vertex of degree 4. In fact, the outdegree sequence of G is completely determined.

Claim: G has exactly 4 vertices of indegree 3 and 4 vertices of indegree 4.

Let $v \in V(G)$. Since both $G \restriction N_{in}(v)$ and $G \restriction N_{out}(v)$ are 1-e.c., it follows that $3 \leq |N_{in}(v)| \leq 4$. Let x be the number of vertices of indegree 3, and let y be the number of vertices of indegree 4. Then since the sum of all indegrees is the number of arcs,

$$\begin{aligned} x+y &= 8 \\ 3x+4y &= 28. \end{aligned}$$

Solving the system establishes the claim.

Now suppose $V(G) = \{1, \ldots, 8\}$. For each vertex v of G, one of the subgraphs induced by $N_{in}(v)$ and $N_{out}(v)$ is D_3 and the other is the tournament of Figure 4.

Without loss of generality, suppose vertex 1 has indegree 4 and the subgraphs induced by $N_{in}(1)$ and $N_{out}(1)$ are as in Figure 4.

Case 1: Vertex 8 has indegree 3.

Without loss of generality, by the symmetry of 2, 3, and 4 in the directed graph in Figure 7, $(2,8),(3,8),(8,4) \in E(G)$. $N_{in}(8) = \{2,3,7\}$ and $(2,3) \in E(G)$, so for $G \restriction N_{in}(8) \cong D_3$, also $(3,7),(7,2) \in E(G)$. $N_{out}(8) = \{1,4,5,6\}$ and $(5,1),(5,6) \in E(G)$, so $(4,5) \in E(G)$.

Now $|N_{out}(7)| = 4$, so all remaining arcs meeting 7 must be directed toward 7, so $(4,7) \in E(G)$. Then $N_{in}(7) = \{3,4,6\}$ and $(3,4) \in E(G)$, so $(4,6),(6,3) \in E(G)$. The vertices 4, 5, and 8 are in $N_{in}(6)$, but $(8,5),(4,5) \in E(G)$ so $|N_{in}(6)| = 4$, so $(2,6) \in E(G)$. Now $N_{in}(6) = \{2,4,5,8\}$ and $(4,5),(8,5) \in E(G)$ so $(5,2) \in E(G)$.

Now we have all but one arc of G, either $(3,5)$ or $(5,3)$. See Figure 5. If that arc were $(3,5)$, then $N_{out}(3) = \{4,5,7,8\}$ and $(4,5),(7,5),(8,5) \in$

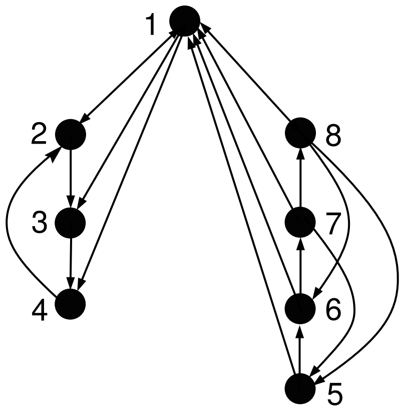

FIGURE 4. The in- and out-neighbours of 1.

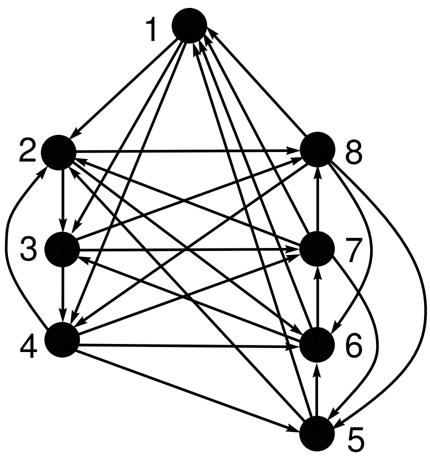

FIGURE 5. G missing one arc.

$E(G)$ which is a contradiction. Otherwise, if that arc were $(5,3)$, then $N_{out}(5) = \{1,2,3,6\}$ and $(1,3),(2,3),(6,3) \in E(G)$, which is a contradiction.

Case 2: Vertex 8 has indegree 4.

In this case $(2,8),(3,8),(4,8) \in E(G)$. Then $N_{out}(8) = \{1,5,6\}$, but $(5,1),(6,1) \in E(G)$, which is a contradiction. □

To find 2-e.c. tournaments of all possible even orders as described in Theorem 3, it is sufficient to give an example of a 2-e.c. tournament of

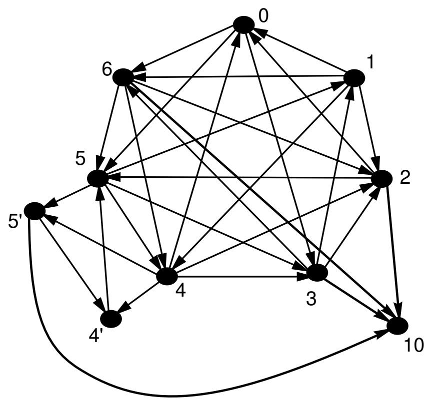

FIGURE 6. The tournament R'. Reverse the arc $(2,1)$ in $R(D_7,(5,4))$ (where $(4,4')$ and $(5,5')$ are arcs), and add a new vertex 10 so that $N_{in}(10)=\{2,3,5',6\}$. Note that not all arcs are shown.

order 10, and then use replication. For this, see the tournament R' in Figure 6. It is straightforward to verify that R' is 2-e.c.: one need only check the vertices 1, 2, and 10 versus each of the other vertices. The details are tedious and are therefore omitted.

In [10] it was proved that whenever there is a 2-e.c. graph of order m, then there is an 2-e.c. graph of order $m+1$, and the question of this type of monotonicity was raised in general for n-e.c. graphs. We remark that the "gap" for 2-e.c. tournaments supplies the first example of non-monotonicity of a 2-e.c. property.

4. Examples of 2-e.c. critical tournaments

Definition 5. *An arc $e=(a,b)$ of tournament G is **good** if every vertex $v\neq a,b$ is the unique solution to some 2-e.c. tournament problem not involving a or b.*

Lemma 7. *Let G be a 2-e.c. critical tournament and let arc $e=(a,b)$ be good. Then each replicate $R=R(G,e)$ is a 2-e.c. critical tournament.*

Proof. Note that: the unique solution of $\begin{pmatrix} b & b' \\ \uparrow & \downarrow \end{pmatrix}$ is a, of $\begin{pmatrix} b & b' \\ \downarrow & \uparrow \end{pmatrix}$ is a, of $\begin{pmatrix} a & a' \\ \uparrow & \downarrow \end{pmatrix}$ is b', and of $\begin{pmatrix} a & a' \\ \downarrow & \uparrow \end{pmatrix}$ is b. Now let $x\in V(R)-\{a,a',b,b'\}$. By hypothesis, x is the unique solution to some 2-e.c.

tournament problem in G. If a' were a solution to this problem in R then a would be a solution to it in G, and if b' were a solution to this problem in R, then b would be a solution to it in G. Therefore, x is the unique solution to this problem in R. □

Definition 6. *In the definition of replication of the arc $e = (a, b)$ in tournament G, we insist that the arc between a and a' be (a, a'), and the arc between b and b' be (b', b), then we call the replication a* type-1 replication*, and use a subscript* 1 *to indicate the resulting tournament, $R_1(G, e)$.*

Lemma 8. *Let G be a 2-e.c. critical tournament and let $e = (a, b) \in E(G)$ be good. Repeatedly replicating e using type-1 replication gives a 2-e.c. critical tournament.*

Proof. Define $G_0 = G$. For $k \geq 0$, define $G_{k+1} = R_1(G_k, e)$, and call the replication arc $e_{k+1} = (a_{k+1}, b_{k+1})$. Then by Lemma 5, G_{k+1} is a 2-e.c. tournament of order $|V(G)| + 2k$. We need to show that G_{k+1} is 2-e.c. critical.

We proceed by induction on k. Assume G_k is 2-e.c. critical and that for $1 \leq j \leq k$, vertex a_j uniquely solves $\begin{pmatrix} b_j & b \\ \uparrow & \downarrow \end{pmatrix}$ and vertex b_j uniquely solves $\begin{pmatrix} a_j & a \\ \downarrow & \uparrow \end{pmatrix}$. Consider G_{k+1}.

Since (a, b) is good in G, each vertex $v \in V(G)\backslash\{a, b\}$ is the unique solution to some 2-e.c. tournament problem in G not involving a or b. In G_{k+1}, no vertex a_j or b_j, $1 \leq j \leq k + 1$ can solve this problem, because otherwise, a or b would have solved it in G.

Vertices a_{k+1} and b_{k+1} can not solve the problems that a_j and b_j $(1 \leq j \leq k)$ uniquely solve in the induction hypothesis: for the a_j problem, (b_j, a_{k+1}) and (b_{k+1}, b) are arcs of G_{k+1}; for the b_j problem, (b_{k+1}, a_j) and (a, a_{k+1}) are arcs of G_{k+1}.

Vertex a_{k+1} uniquely solves $\begin{pmatrix} b_{k+1} & b \\ \uparrow & \downarrow \end{pmatrix}$ since every vertex except a_{k+1} and a (and b) is directed the same way with respect to b and b_{k+1}, and a is directed toward b. By a similar argument, vertex b_{k+1} uniquely solves $\begin{pmatrix} a_{k+1} & a \\ \downarrow & \uparrow \end{pmatrix}$, since every vertex except b_{k+1} and b (and a) is directed the same way with respect to a and a_{k+1}, and a is directed toward b.

Finally, a uniquely solves $\begin{pmatrix} b_{k+1} & b \\ \downarrow & \uparrow \end{pmatrix}$ and b uniquely solves $\begin{pmatrix} a_{k+1} & a \\ \uparrow & \downarrow \end{pmatrix}$. □

Using Lemmas 7 and 8, we may construct 2-e.c. critical tournament for all the possible orders k, where $k \geq 7$ and $k \neq 8$ as follows. For the odd orders, we note that in D_7, $(4, 3)$ is a good arc, as demonstrated by the following table.

vertex	0	1	2	5	6
uniquely solves	$\begin{pmatrix} 2 & 5 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ \uparrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ \uparrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ \downarrow & \downarrow \end{pmatrix}$

For the even orders, the following tables demonstrate that the tournament R' (introduced at the end of Section 3) is 2-e.c. critical, and that $(0, 6)$ is a good arc.

vertex	0	1	2	3	4
uniquely solves	$\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5 & 4 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 4 & 3 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5' & 4' \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5 & 5' \\ \downarrow & \uparrow \end{pmatrix}$

vertex	5	6	$4'$	$5'$	10
uniquely solves	$\begin{pmatrix} 4 & 4' \\ \uparrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5' & 3 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 4 & 4' \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 2 & 5' \\ \downarrow & \downarrow \end{pmatrix}$

We close with the following problem: find examples of n-e.c. tournaments, where $n \geq 3$, that are not Paley tournaments.

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