INFINITE GEOMETRIC GRAPHS AND PROPERTIES OF METRICS

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Abstract. We consider isomorphism properties of infinite random geometric graphs defined over a variety of metrics. In previous work, it was shown that for $\mathbb{R}^n$ with the L_∞-metric, the infinite random geometric graph is, with probability 1, unique up to isomorphism. However, in the case $n = 2$ this is false with either of the L_2-metric or the hexagonal metric. We generalize this result to a large family of metrics induced by norms. In particular, we show that the infinite geometric graph is unique up to isomorphism if and only if the metric space has a new property which we name truncating: each step-isometry from a dense set to itself is an isometry. As a corollary, we derive that the infinite random geometric graph defined in L_p space is unique up to isomorphism with probability 1 only in the cases when $p = 1$ or $p = \infty$.

1. Introduction

Geometric random graph models play an important role in the modelling of real-world networks such as on-line social networks [6], wireless networks [17, 19], and the web graph [1, 16]. In such stochastic models, vertices of the network are represented by points in a suitably chosen metric space, and edges are chosen by a mixture of relative proximity of the vertices and probabilistic rules. In real-world networks, the underlying metric space is a representation of the hidden reality that leads to the formation of edges. Such networks can be viewed as embedded in a *feature space*, where vertices with similar features are more closely positioned. For example, in the case of on-line social networks, for example, users are embedded in a high dimensional *social space*, where users that are positioned close together in the space exhibit similar characteristics. The web graph may be viewed in *topic space*, where web pages with similar topics are closer to each other. We note that the theory of random geometric graphs has been extensively developed (see, for example, [2, 14] and the book [22]).

The study of countably infinite graphs is motivated in part by the theory of the *infinite random graph*, or the *Rado* graph (see [10, 11, 15]), written R. The graph R was first discovered by Erdős and Rényi [15], who proved that with probability 1, any two randomly generated countably infinite graphs, where vertices are joined independently with probability $p \in (0, 1)$, are isomorphic. The graph R has several remarkable properties, such as universality (all countable graphs are isomorphic to an induced subgraph) and homogeneity (every isomorphism between finite induced subgraphs extends to an isomorphism). The investigation of R lies at the intersection of logic, probability theory, and topology; see the surveys [10, 11, 15] and Chapter 6 of [5].

Geometric random graphs are usually studied in the case of finite graphs. In [7] we considered infinite random geometric graphs. In our model, vertices were chosen at random according to a given probability distribution from a dense subset of $\mathbb{R}^n$ for a fixed $n \geq 1$ and

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two vertices are adjacent if the distance between the two vertices is no larger than some fixed real number. More precisely, consider a metric space S with metric

$$d : S \times S \to \mathbb{R},$$

a parameter $\delta \in \mathbb{R}^+$, a countably infinite subset V of S, and $p \in (0, 1)$. The *Local Area Random Graph* LARG(V, δ, p) has vertices V, and for each pair of vertices u and v with $d(u, v) < \delta$, an edge is added independently with probability p. Note that V may be either finite or infinite. For simplicity, we consider only the case when $\delta = 1$; we write LARG(V, p) in this case. The LARG model generalizes well-known classes of random graphs. For example, special cases of the LARG model include the random geometric graphs, where $p = 1$, and the binomial random graph $G(n, p)$, where S has finite diameter D, and $\delta \geq D$.

A set $V \subseteq \mathbb{R}^2$ is *representative* if, for every set $A \subseteq \mathbb{R}^2$ with positive Lebesgue measure, $A \cap V \neq \emptyset$. It follows that a representative set must be infinite, dense in $\mathbb{R}^2$, and that the intersection with any set of non-zero measure must be infinite. For example, in $\mathbb{R}^2$ with the Euclidean L_2-metric, the set of points with irrational coordinates is representative, but the set of points with rational coordinates is not. The concept of representative set is introduced to capture the notion of a "random" countably infinite subset of $\mathbb{R}^2$. Thus, if an infinite number of points from $\mathbb{R}^2$ are chosen according to, for example, a Poisson point process with uniform density, then the result would, with probability 1, be a representative set.

Let Ω_n be the set of metrics defined on $\mathbb{R}^n$. We will say that a metric $d \in \Omega_n$ is *predictable* if there exists a countably infinite representative set $V \subseteq \mathbb{R}^n$ such that for all $p \in (0, 1)$, with probability 1 graphs generated by LARG(V, p) are isomorphic. For all $n \in \mathbb{N}^+$ it was shown in [7] that the L_∞-metric is predictable, while the L_2-metric is not.

We have therefore, the following natural classification program for infinite random geometric graphs.

Geometric Isomorphism Dichotomy (GID): Determine which metrics in Ω_n are predictable.

In the next section, we describe our main results which greatly extend our understanding of the GID for a large family of so-called norm derived metrics, which includes all the familiar L_p-metrics and the hexagonal metric. The hexagonal metric arises in the study of Voronoi diagrams and period graphs (see [18]) which have found applications to nanotechnology. We note that in [8], it was shown that the hexagonal metrics in the case $n = 2$ are not predictable.

All graphs considered are simple, undirected, and countable unless otherwise stated. Given a metric space S with distance function d, define the (open) *ball of radius δ around x* by

$$B_\delta(x) = \{u \in S : d(u, x) < \delta\}.$$

We will sometimes just refer to $B_\delta(x)$ as a *δ-ball* or *ball of radius δ*. A subset V is *dense* in S if for every point $x \in S$, every ball around x contains at least one point from V. We refer to $u \in S$ as points or vertices, depending on the context. Throughout, let $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}$, and $\mathbb{R}$ denote the non-negative integers, the positive integers, the integers, and real numbers, respectively. We use **bold** notation for vectors $\mathbf{u} \in \mathbb{R}^n$. The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u} \cdot \mathbf{v}$. For a reference on graph theory the reader is directed to [13, 24], while [9] is a reference on metric spaces.

1.1. **Main results.** Our main result is Corollary 3 stated below, which settles the GID for a large class of metrics in the plane. Before we state the theorem, we first need some definitions. *Throughout, d is a metric in Ω_2.* Indeed, the definitions stated in this section naturally extend to higher dimensions and more general metric spaces, but we focus on the plane for simplicity, since our results apply there.

1.1.1. *The truncation property of metric spaces.* A function $f : V \to V$, where $V \subseteq \mathbb{R}^2$ is a *step-isometry* if for every $u, v \in V$,

$$\lfloor d(u,v) \rfloor = \lfloor d(f(u), f(v)) \rfloor.$$

Thus, a step-isometry preserves distances in truncated form. A metric d has the *truncating property* if every step-isometry $f : V \to W$ is an isometry, where $V \subseteq \mathbb{R}^2$ is a representative set. In [7], it was shown that $\mathbb{R}^2$ with the L_2-metric has the truncating property, but the L_∞-metric does not. For example, in the case $n = 1$ with the L_∞-metric, consider the map $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \lfloor x \rfloor + \frac{2}{3}(x - \lfloor x \rfloor) & \text{if } x - \lfloor x \rfloor \leq \frac{1}{2}, \\ \lfloor x \rfloor + \frac{4}{3}(x - \lfloor x \rfloor) - \frac{1}{3} & \text{else.} \end{cases}$$

It is straightforward to check that f is a step-isometry, but not an isometry.

1.1.2. *Norm-derived metrics and their shape.* A metric d is *translation-invariant* if for all $\mathbf{a}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, we have that

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x} + \mathbf{a}, \mathbf{x} + \mathbf{a}).$$

The metric d is *homogeneous* if for all $\alpha \in \mathbb{R}$, $d(\alpha\mathbf{x}, \alpha\mathbf{y}) = |\alpha| d(\mathbf{x}, \mathbf{y})$. Any translation-invariant, homogeneous metric d induces a norm, given by $\|\mathbf{x}\| = d(\mathbf{x}, \mathbf{0})$. Conversely, a norm $\|\cdot\|$ induces a translation-invariant, homogeneous metric

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Hence, the metrics we consider are precisely those that are derived from norms. We define a *norm-derived* metric as a metric defined by a norm (referred to as the *underlying norm*). A well-studied family of norm-derived metrics consists of metrics derived from the L_p norm, where $p \geq 1$; recall that for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p}$.

It is straightforward to check that the unit ball around $\mathbf{0}$ of any norm-derived metric d must be a convex, point-symmetric set P which we call the *shape* of d. Conversely, a convex, point-symmetric set $P \subseteq \mathbb{R}^2$ defines a norm $\|\cdot\|_P$ and a corresponding metric d_P as follows. Fix a vector $\mathbf{x} \in \mathbb{R}^2$, and let $\mathbf{a}$ be the unique point where the ray from $\mathbf{0}$ to $\mathbf{x}$ intersects P. Then we have that

$$\|\mathbf{x}\|_P = \frac{\|\mathbf{a}\|_2}{\|\mathbf{x}\|_2},$$

and the metric d_P is defined as

$$d_P(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_P.$$

Observe that the unit ball around $\mathbf{0}$ of d_P equals P. Note that in $\mathbb{R}^2$, for $p > 1$, the L_p-metric has shape a superellipse or Láme curve, while the L_∞-metric has shape a square with sides parallel to the coordinate axes, and the L_1-metric has shape a square with the diagonals parallel to the coordinate axes. Throughout this paper, we will use the notation d_P for the norm-derived metric with shape P.

We only consider norm-derived metrics whose shape is either a polygon (and we call such metrics *polygonal*; these include the L_p-metrics in the case $p = 1, \infty$), or metrics whose shape can be described by a smooth curve (we call such metrics *smooth*; which includes the L_p-metrics in the case $p > 1$ and $p \neq \infty$). We think that the results in this paper apply equally to norm-based metric whose shape has both straight and curved sides, and that this can be proved with methods similar to those presented here. However, we do not pursue this generalization here, due to the excessive technicalities. *To conserve notation, from now on we denote by* Ω *the set of all polygonal and smooth norm-derived metrics in* $\mathbb{R}^2$.

1.1.3. *Main results.* We now have defined almost all concepts needed to state our main result, which constitutes a classification of all metrics in Ω in terms of the truncating property. We next define a special class of polygonal metrics; our main result will show that this is the class of metrics in Ω which do not have the truncating property. A metric $d \in \Omega$ is a *box metric* if and only if its shape is a *parallelogram*.

Theorem 1. *A metric* $d \in \Omega$ *has the truncating property if and only if* d *is not a box metric.*

The proof of the theorem will follow from a series of lemmas presented in Section 2. The proof considers separately the polygonal and smooth cases.

An important corollary of Theorem 1 shows that, in $(\mathbb{R}^2, d_P)$ where d_P is not a box metric, three well-chosen points completely determine a step-isometry, as they do an isometry. For example, if d_P is the L_2-metric, then the restriction on the 3 points is that they should not be collinear. For general norm-derived metrics, the restriction is slightly more complicated, and we need some definitions before we can state the corollary.

A set of three points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ so that $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{y}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{z})$ is a *triangular set* if

$$d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) > d(\mathbf{x}, \mathbf{z}).$$

If d is the L_2-metric, then $\mathbf{x}, \mathbf{y}, \mathbf{z}$ forms a triangular set if and only if the points are not collinear; the same is not true for metrics where the unit ball P is a polygon. For example, let d_∞ be the metric derived from the L_∞-norm, and let $\mathbf{x} = (0, 0)$, $\mathbf{y} = (2, 0)$, and $\mathbf{z} = (1, 1)$. Then $d_\infty(\mathbf{x}, \mathbf{z}) = d_\infty(\mathbf{y}, \mathbf{z}) = 1$ and $d_\infty(\mathbf{x}, \mathbf{y}) = 2$, so $\mathbf{x}, \mathbf{y}, \mathbf{z}$ do not form a triangular set, despite the fact they are not collinear.

Triangular sets exist in any representative set and for any metric $d \in \Omega$ which is not a box metric. Namely, fix $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ such that the lines through $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{y}$ and $\mathbf{z}$, and $\mathbf{z}$ and $\mathbf{x}$ have slope γ_1, γ_2, γ_3, respectively. If d is a smooth metric, then if γ_1, γ_2, γ_3 are all different and none of them is parallel to the x- or y-axis, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a triangular set. If d is a polygonal metric, and if the lines through the origin with slopes γ_1, γ_2, γ_3 intersect P in three different, non-parallel sides, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a triangular set. It is clear that such sets exist in any set that is representative and therefore, dense in $\mathbb{R}^2$, and d is not a box metric.

It follows immediately from the properties of norm-derived metrics that triangular sets have the following *anchoring property*: if V is a representative set, $S \subseteq V$ is a triangular set, and $f : V \to V$ an isometry, then the images under f of the points in S completely determine the map f. Hence, images of triangular sets completely determine the images of points of a representative set under an isometry. Thus, it follows as corollary from Theorem 1 that if d is not a box metric, then triangular sets have the anchoring property if only the truncated distances are given.

Corollary 2. *Let V be a representative set and let $S \subseteq V$ be a triangular set, and let $d \in \Omega$ be a norm-based metric which is not a box metric. Then for every step-isometry $f : V \to V$, the images under f of the points in S completely determine f.*

Proof. By Theorem 1, the step-isometry f is an isometry. The points of S completely determine, therefore, the position of all points in V. The proof now follows since the image of a triangular set under an isometry is also a triangular set. □

Thus, if the coordinates of the points in S are given, and for all other points in V the truncated distance to each of the points in S is given, then the coordinates of all points in V are determined.

Arguments involving infinite graphs can be used with Corollary 2 to prove the following theorem, which settles the GID for all metrics in Ω, including the L_p-metrics and polygonal metrics on $\mathbb{R}^2$.

Theorem 3. *A metric $d \in \Omega$ is predictable if and only if it is a box metric.*

We defer the proof to the end of Section 2. We conjecture that analogous results as in Theorem 3 apply to higher dimensions and other norm-derived metrics. We will consider these cases in future work.

2. Outline of the proof of the main results

In this section, we will sketch the proof of Theorem 1, and give the sequence of lemmas needed to arrive at the proof. We will also give the proof of Theorem 3. Before we proceed to an outline of the proof of Theorem 1, we first introduce some notation and concepts that apply to all norm-derived metrics, and will be helpful in the rest of the paper. For a polygonal metric $d_P \in \Omega$, we can use the description of the polygon to compute the norm and distance. We introduce specific notation for this case which we will use throughout. Let P be a point-symmetric (but possible non-regular) polygon in $\mathbb{R}^2$ whose boundaries are formed by lines with normals $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$. See Figure 1 for an example. More precisely, let P be the set:

$$P = \{\mathbf{x} : \text{for all } 1 \le i \le k, \ -1 \le \mathbf{a}_i \cdot \mathbf{x} \le 1\}.$$

Then for each $\mathbf{x} \in \mathbb{R}^2$, we have that:

$$\|\mathbf{x}\|_P = \max_{1 \le i \le k} |\mathbf{a}_i \cdot \mathbf{x}|.$$

For $\mathbf{a} \in \mathcal{G}$, let $F(P, \mathbf{a})$ be the face of P with normal $\mathbf{a}$, where $\mathbf{a}$ is pointing away from the centre of P. Define $F(P, -\mathbf{a})$ analogously. See Figure 1.

Define the set of *generators* of P, written $\mathcal{G}_P$, to be the set of vectors that define its sides; in particular, we have that

$$\mathcal{G}_P = \{\mathbf{a}_i : 1 \le i \le k\} \cup \{-\mathbf{a}_i : 1 \le i \le k\}.$$

Observe that an alternative description is $P = \{\mathbf{x} : \text{for all } \mathbf{a} \in \mathcal{G}_P, \ 0 \le \mathbf{a} \cdot \mathbf{x} \le 1\}$, and for each $\mathbf{x} \in \mathbb{R}^n$, we have that

$$\|\mathbf{x}\|_P = \max_{\mathbf{a} \in \mathcal{G}_P} |\mathbf{a} \cdot \mathbf{x}|.$$

We now consider the case where the metric is smooth. Let $\mathcal{G}_P$ be the set of all vectors from the origin to points on the boundary of P. Now let $\mathcal{G}_P^*$ be a countable dense subset of $\mathcal{G}_P$. In

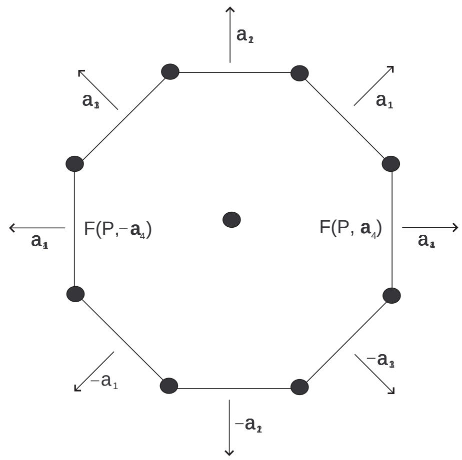

FIGURE 1. The polygon P.

particular, for each vector $\mathbf{a} \in \mathcal{G}_P$ and all $\epsilon > 0$, we require that there exist $\mathbf{b} \in \mathcal{G}_P^*$ so that $\mathbf{a} \cdot \mathbf{b} \geq 1 - \epsilon$. In other words, the angle between $\mathbf{a}$ and $\mathbf{b}$ can be made arbitrarily small. Let

$$P^* = \{\mathbf{x} : \text{for all } \mathbf{a} \in \mathcal{G}_P^*,\ 0 \leq \mathbf{a} \cdot \mathbf{x} \leq 1\}.$$

Then the closure of P^* is P. Moreover, we have that for all $\mathbf{x} \in \mathbb{R}^2$,

$$\|\mathbf{x}\|_P = \sup_{\mathbf{a} \in \mathcal{G}_P^*} \mathbf{a} \cdot \mathbf{x}.$$

Such a set $\mathcal{G}_P^*$ will be called a *generator set* for P. Note that every convex set has a countable generator set, while only polygons have finite generator sets.

The proof of Theorem 1 is outlined here through a series of lemmas. The proofs of these technical lemmas can all be found in Sections 3 and 4. Their proofs are based on the concept of respected lines, which we now introduce. Let $V \subseteq \mathbb{R}^2$ and $f : V \to \mathbb{R}^2$ be an injective map. Let $r \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^2$, and let ℓ be the line defined by the equation $\mathbf{a} \cdot \mathbf{x} = r$. The map f *respects* the line ℓ if there exists a line ℓ' with equation $\mathbf{a}' \cdot \mathbf{x} = r'$ for some $r' \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^2$ such that for all $\mathbf{v} \in V$

$$\mathbf{a} \cdot \mathbf{v} < r \text{ implies that } \mathbf{a}' \cdot f(\mathbf{v}) \leq r', \text{ and } \mathbf{a} \cdot \mathbf{v} > r \text{ implies that } \mathbf{a}' \cdot f(\mathbf{v}) \geq r'. \tag{1}$$

The line ℓ' will be called the *image* of ℓ under the *line map*. Note that a function f respects a line ℓ if the half-spaces on both sides of ℓ are mapped to half-spaces separated by another line, which can be then be considered the image of ℓ. An *integer parallel* of line ℓ is a line ℓ' parallel to ℓ so that $d(\ell, \ell') \in \mathbb{Z}$ (the distance between two lines is defined in the obvious way).

The first lemma establishes some straightforward consequences of the definitions.

Lemma 4. *Let V be a representative set in $\mathbb{R}^2$ and let $f : V \to V$ be a bijection. Suppose f respects the line ℓ, and ℓ' is the image of ℓ under the line map.*

(i) If ℓ contains a point $\mathbf{v} \in V$, then ℓ' contains $f(\mathbf{v})$.
(ii) The line ℓ' is the unique image of ℓ under the line map. That is, ℓ' is the only line which satisfies (1).

(*iii*) *Suppose f respects the line $\hat{\ell}$, and $\hat{\ell}'$ is the image of $\hat{\ell}$ under the line map. Then ℓ and $\hat{\ell}$ are parallel if and only if ℓ' and $\hat{\ell}'$ are parallel.*

(*iv*) *If f is a step-isometry and ℓ is a line respected by f, then all integer parallels of ℓ must also be respected.*

The first step in the proof of the main result is to establish that there are some lines that must be respected by any step-isometry on a representative set. We consider the smooth case first.

Lemma 5. *Consider $\mathbb{R}^2$ equipped with a smooth metric $d \in \Omega$. Let V be a representative set in $\mathbb{R}^2$, and let $f : V \to V$ be a step-isometry. Let $\mathbf{v}_1, \mathbf{v}_2 \in V$, and let ℓ be the line through $\mathbf{v}_1$ and $\mathbf{v}_2$. Then ℓ must be respected, and its image is the line ℓ' through $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$. Moreover, any line through a point $\mathbf{v}_3 \in V$ and parallel to ℓ must be respected, and its image is the line through $f(\mathbf{v}_3)$ and parallel to ℓ'.*

A similar lemma for the polygonal case involves a more technical argument. The proof of this lemma can be found in Section 3.

Lemma 6. *Consider $\mathbb{R}^2$ equipped by a polygonal metric $d_P \in \Omega$. Let V be a representative set in $\mathbb{R}^2$ and let $f : V \to V$ be a step-isometry. Then any line through a point $\mathbf{v} \in V$ and parallel to one of the sides of P must be respected, and its image is a line through $f(\mathbf{v})$ parallel to one of the sides of P.*

The proof of Theorem 1 is based on the fact that the lines emanating from a finite number of points generate a grid which is infinitely dense, which means that these points completely determine the step-isometry. These ideas will be made precise in Lemma 8 stated below.

We introduce some terminology which describes the lines that form the dense grid of lines. Let B be a finite set of points in $\mathbb{R}^2$, and let $\mathcal{G}$ be a set of vectors in $\mathbb{R}^2$. Define a collection of lines

$$\mathcal{L}(B, \mathcal{G}) = \bigcup_{i \in \mathbb{N}} \mathcal{L}_i(B, \mathcal{G}).$$

inductively as follows. The set $\mathcal{L}_0(B, \mathcal{G})$ contains all lines through points in B with normal vector in $\mathcal{G}$, as well as their integer parallels. For $i > 0$, the sets $\mathcal{L}_i(B, \mathcal{G})$ are defined inductively. Assume that $\mathcal{L}_i(B, \mathcal{G})$ has been defined for some $i \geq 0$. Then $\mathcal{L}_{i+1}(B, \mathcal{G})$ consists of all lines with normal vector in $\mathcal{G}$ going through a point $\mathbf{p}$ which is an intersection point of two lines in $\mathcal{L}_i(B)$. See Figure 2.

Now let V be a representative set in $\mathbb{R}^2$, and $f : V \to V$ a step-isometry. Lemmas 5 and 6 can be used to show that, given a set $B \subseteq V$, there exists a countable generator set $\mathcal{G}$ of P so that $\mathcal{L}_0(B, \mathcal{G})$ must be respected by f. Using an inductive argument, it can be shown that, if all lines in $\mathcal{L}_0(B, \mathcal{G})$ must be respected, then all lines in $\mathcal{L}(B, \mathcal{G})$ must be respected. This results in the following crucial lemma.

Lemma 7. *Let $d_P \in \Omega$, let V be a representative set in $\mathbb{R}^2$, and let $f : V \to V$ be a step-isometry. Then there exists a countable generator set $\mathcal{G}$ for P, so that for each set $B \subseteq V$, f must respect all lines in $\mathcal{L}(B, \mathcal{G})$.*

Moreover, there exists a map $\sigma : \mathcal{G} \to \mathbb{R}^2$ such that, if a line $\ell \in \mathcal{L}(B, \mathcal{G})$ has normal vector $\mathbf{a} \in \mathcal{G}$ then its image under the line map has normal vector $\sigma(\mathbf{a})$. In addition, the set $\mathcal{G}' = \{\sigma(\mathbf{a}) : \mathbf{a} \in \mathcal{G}\}$ is a generator set for P.

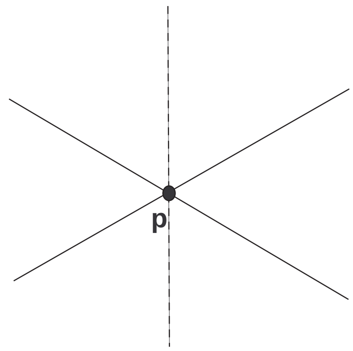

FIGURE 2. The solid lines are in $\mathcal{L}_i(B,\mathcal{G})$ and the dotted line is in $\mathcal{L}_{i+1}(B,\mathcal{G})$.

If the vectors in $\mathcal{G}$ fall into two parallel classes, then $\mathcal{L}(B,\mathcal{G}) = \mathcal{L}_0(B,\mathcal{G})$. In particular, in this case no new lines are generated in the induction step. The following lemma shows that, if d_P is not a box metric, and thus, any generator set $\mathcal{G}$ for P contains at least three vectors that are pairwise non-parallel, then B can be chosen so that $\mathcal{L}(B,\mathcal{G})$ is dense in $\mathbb{R}^2$. Thus, the lines in $\mathcal{L}(B,\mathcal{G})$ generate a dense grid, and any step-isometry must be consistent with this grid.

Lemma 8. *Let $d_P \in \Omega$, and let $\mathcal{G}$ be a countable generator set for P which contains at least three vectors that are pairwise non-parallel. Let $B = \{\mathbf{p},\mathbf{q}\}$, and fix $\mathbf{a} \in \mathcal{G}$, where $\mathbf{a}\cdot(\mathbf{p}-\mathbf{q}) = r \in (0,1)$. Then the family $\mathcal{L}(B,\mathcal{G})$ contains all the following lines, where $z_1, z_2 \in \mathbb{Z}$:*

$$\mathbf{a}\cdot(\mathbf{x}-\mathbf{q}) = z_1 r + z_2.$$

Moreover, if r is irrational, then the lines in $\mathcal{L}(B,\mathcal{G})$ form a dense grid; that is, the set of values $\{\mathbf{a}\cdot(\mathbf{x}-\mathbf{q}) : \mathbf{x} \in \mathbb{R}^2\}$ is dense in $\mathbb{R}$.

The final lemma leads to the proof of the main result. It shows that any step isometry on a representative set gives rise to an isometry on a dense set of points in $\mathbb{R}^2$ formed by a grid of lines which all have to be respected.

Lemma 9. *Let $d_P \in \Omega$ be a norm-based metric which is not a box metric, and let $\mathcal{G}$ be a generator set for P so that Lemma 7 holds. Let V be a representative set in $\mathbb{R}^2$, and let $f : V \to V$ be a step-isometry. Then for every $\mathbf{q} \in V$, there exists a point $\mathbf{p} \in V$ such that, for all $\mathbf{a} \in \mathcal{G}$, $\mathbf{a}\cdot(\mathbf{p}-\mathbf{q})$ is irrational. Moreover, let $B = \{\mathbf{p},\mathbf{q}\}$, and let $\sigma : \mathcal{G} \to \mathbb{R}^2$ be as defined in Lemma 7. Then for each $\mathbf{a} \in \mathcal{G}$ there exists a set $R(\mathbf{a})$ which is dense in $\mathbb{R}$, and for which the following holds: For each $r \in R(\mathbf{a})$, the line ℓ with equation $\mathbf{a}\cdot(\mathbf{x}-\mathbf{q}) = r$ must be respected, and the image of ℓ has equation $\sigma(\mathbf{a})\cdot(\mathbf{x}-\mathbf{q}') = r$, where $\mathbf{q}' = f(\mathbf{q})$.*

With these lemmas at our disposal, we may now supply a proof of the main theorem.

Proof of Theorem 1. For the forward direction, assume that $d_P \in \Omega$ is a box metric. Thus, P is a parallelogram, and d_P has a set of two non-parallel generators $\mathbf{a}_1$ and $\mathbf{a}_2$, so that

$$P = \{\mathbf{x} : -1 \le \mathbf{a}_i \cdot \mathbf{x} \le 1 \text{ for } i = 1,2\}.$$

Now consider the invertible linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which sends $\mathbf{a}_1$ to $(1,0)^T$ and $\mathbf{a}_2$ to $(0,1)^T$. Then for all $\mathbf{u},\mathbf{v} \in \mathbb{R}^2$, $d_P(\mathbf{u},\mathbf{v}) = d_\infty(T(\mathbf{u}),T(\mathbf{v}))$. In [7], it was shown that, for $\mathbb{R}^n$ with the L_∞ metric, there are uncountably many step isometries on $\mathbb{R}^2$ which are not isometries, and that, for any representative set V and $p \in (0,1)$, with probability 1 two graphs produced by LARG(V,p) are isomorphic. Using this result and the transformation T, we can see that the same is true for any box metric.

For the reverse direction, let V be a representative set and f a step-isometry; we must show that f is in fact an isometry. Let $d_P \in \Omega$ be a norm-based metric which is not a box metric. Let $\mathcal{G}$ be a countable generator set for P for which Lemma 7 holds.

Fix $\mathbf{q}, \mathbf{v} \in V$, and $\mathbf{a} \in \mathcal{G}$; for a point $\mathbf{x}$ we use the notation $\mathbf{x}' = f(\mathbf{x})$. Choose $\mathbf{p} \in V$ so that Lemma 9 holds, and let $B = \{\mathbf{p}, \mathbf{q}\}$. Let $\sigma : \mathcal{G} \to \mathbb{R}^2$ be as given by Lemma 7. Let $R(\mathbf{a})$ be the set as given by Lemma 9.

Assume that $\mathbf{a} \cdot (\mathbf{v} - \mathbf{q}) < \sigma(\mathbf{a}) \cdot (\mathbf{v}' - \mathbf{q}')$. Let $t \in R(\mathbf{a})$ be such that

$$\mathbf{a} \cdot (\mathbf{v} - \mathbf{q}) < t < \sigma(\mathbf{a}) \cdot (\mathbf{v}' - \mathbf{q}').$$

By Lemma 9, the line ℓ with equation $\mathbf{a} \cdot (\mathbf{x} - \mathbf{q}) = t$ is in $\mathcal{L}(B, \mathcal{G})$, and thus must be respected by f, and the image of ℓ is the line ℓ' with equation $\sigma(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{q}') = t$. Thus, $\mathbf{v}$ lies to the left of ℓ, while $\mathbf{v}'$ lies to the right of ℓ'. This is a contradiction.

An analogous argument shows that the assumption $\mathbf{a} \cdot (\mathbf{v} - \mathbf{q}) > \sigma(\mathbf{a}) \cdot (\mathbf{v}' - \mathbf{q}')$ also leads to a contradiction. Thus, we conclude that for all $\mathbf{a} \in \mathcal{G}$:

$$\mathbf{a} \cdot (\mathbf{v} - \mathbf{q}) = \sigma(\mathbf{a}) \cdot (\mathbf{v}' - \mathbf{q}'). \tag{2}$$

Let $\mathcal{G}' = \{\sigma(\mathbf{a}) : \mathbf{a} \in \mathcal{G}\}$. By Lemma 7, $\mathcal{G}'$ is again a generator set for P. By the definition of a generator set, we have that

$$\begin{aligned} d_P(\mathbf{v}, \mathbf{q}) &= \sup\{|\mathbf{a} \cdot (\mathbf{v} - \mathbf{q})| : \mathbf{a} \in \mathcal{G}\} \\ &= \sup\{|\sigma(\mathbf{a}) \cdot (\mathbf{v}' - \mathbf{q}')| : \mathbf{a} \in \mathcal{G}\} \\ &= \sup\{|\mathbf{a} \cdot (\mathbf{v}' - \mathbf{q}')| : \mathbf{a} \in \mathcal{G}'\} \\ &= d_P(\mathbf{v}', \mathbf{q}'), \end{aligned}$$

where the second equality follows by (2). Since $\mathbf{v}, \mathbf{q}$ where arbitrary, f is an isometry. □

We finish the section with the proof of our second main result. For graphs G and H, a *partial isomorphism* from G to H is an isomorphism of some finite induced subgraph of G to an induced subgraph of H. A standard approach to show two countably infinite graphs are isomorphic is to build a chain of partial isomorphisms whose union gives an isomorphism. Using the probabilistic method, we show how this fails for random geometric graphs whose metric is not a box metric. In the following proof, the probability of an event A is denoted by $\mathbb{P}(A)$.

Proof of Theorem 3. Suppose that $\mathbb{R}^2$ is equipped with a metric $d \in \Omega$. Consider graphs produced by the model LARG(V,p) for some representative set V, and real numbers $\delta > 0$ and $p \in (0,1)$.

For the forward direction, assume d is not a box metric. Define an enumeration $\{v_i : i \in \mathbb{N}^+\}$ of V to be *good* if $d(v_i, v_{i+1}) < \delta$ for all $i \in \mathbb{N}^+$ and $\{v_1, v_2, v_3\}$ are not collinear.

Claim 10. *Any representative set V has a good enumeration.*

Proof. For a positive integer n, we call $\{v_i : 1 \leq i \leq n\}$ a *partial good enumeration* of V. We prove the claim by constructing a chain of partial good enumerations by induction. Since V is dense in $\mathbb{R}^2$, we may choose a triangular set whose points are pairwise within δ of each other (see also the discussion in Section 1.1.3 . Let $V_1 = \{v_1, v_2, v_3\}$. Enumerate $V \setminus V_1$ as $\{u_i : i \geq 2\}$. Starting from V_1, we inductively construct a chain of partial good enumerations V_n, $n \geq 1$, so that for $n \geq 2$, V_n contains $\{u_i : 2 \leq i \leq n\}$.

We now want to form V_{n+1} by adding $u = u_{n+1}$. If $u \in V_n$, then let $V_{n+1} = V_n$. Assume without loss of generality that $u \notin V_n$. Let $N = |V_n|$. If $d(v_N, u) < \delta$, then let $v_{N+1} = u$ and add it to V_n to form V_{n+1}. Otherwise, by the density of V, choose a shortest finite path $P = p_0, \ldots, p_\ell$ of points of $V \setminus V_n$ starting at $v_N = p_0$ and ending at $u = p_\ell$ so that two consecutive points in the path are distance at most δ. Then add the vertices of P to V_n to form V_{n+1} and enumerate them so that $v_{N+i} = p_i$ for $i = 0, 1, \ldots, \ell$. Taking the limit of this chain, $\bigcup_{n \geq 1} V_n$ is a good enumeration of V, which proves the claim. □

Let $V = \{v_i : i \geq 1\}$ be a good enumeration of V, and for any n, let $V_n = \{v_i : 1 \leq i \leq n\}$. Let G and H be two graphs produced by LARG(V, p). We say that two pairs $\{v, w\}$ and $\{v', w'\}$ of vertices are *compatible* if $\{v, w\}$ are adjacent in G and $\{v', w'\}$ are adjacent in H or $\{v, w\}$ are non-adjacent in G and $\{v', w'\}$ are non-adjacent in H. For two pairs $\{v, w\}$ and $\{v', w'\}$ such that $d(v, w) = d(v', w')$, the probability that they are compatible equals

$$p^* = \begin{cases} p^2 + (1-p)^2 & \text{if } d(v,w) < \delta, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that G and H are isomorphic, and let f be an isomorphism. Then by Lemma 1 of [8], f must be a step-isometry on V. By Theorem 1, f must be an isometry. By Corollary 2 the images of a triangular set $V_1 = \{v_1, v_2, v_3\}$ in $\mathbb{R}^2$ (we identify these with vertices so do not denote them in bold) determine f completely. Let A_n be the event that there exists a partial isomorphism f from the subgraph induced by V_n into H so that $f(V_1) \subseteq V_n$, and let

$$A_n^* = \bigcap_{\nu \geq n} A_\nu.$$

Note that $A_n^* \subseteq A_{n+1}^*$ for all n.

Next, we estimate the probability of A_n^*. Note first that $\mathbb{P}(A_n^*) \leq \mathbb{P}(A_\nu)$ for all $\nu \geq n$. For any tuple (u_1, u_2, u_3) of distinct vertices in V_n, let $C_n(u_1, u_2, u_3)$ be the event that there exists a partial isomorphism f from the subgraph induced by V_n in G to H so that $f(v_i) = u_i$ for $i = 1, 2, 3$. If C_n happens, then all pairs (v_i, v_{i+1}) and $(f(v_i), f(v_{i+1}))$ must be compatible, for $1 \leq i < n$. Therefore,

$$\mathbb{P}(C_n(u_1, u_2, u_3)) \leq (p^*)^{n-1}.$$

Now

$$A_n = \bigcup_{\{u_1, u_2, u_3\} \subseteq V_n} C_n(u_1, u_2, u_3),$$

so for $n \geq 3$ we have that $\mathbb{P}(A_n) \leq n^{2k+2} (p^*)^{n-1}$, and

$$\mathbb{P}(A_n^*) \leq \inf\{\nu^{2k+2} (p^*)^{\nu-1} : \nu \geq n\} = 0.$$

If B is the event that G and H are isomorphic, then

$$B \subseteq \bigcup_{n \in \mathbb{N}^+} A_n^*.$$

Since the union of countably many sets of measure zero has measure zero, we conclude that $\mathbb{P}(B) = 0$, and thus, with probability 1, G is not isomorphic to H. □

3. Proofs of Lemmas 4, 5, 7, 8 and 9

Proof of Lemma 4. Suppose f respects the line ℓ, and ℓ' is the image of ℓ under the line map. Property (i) follows directly from (1): if ℓ contains a point $\mathbf{v} \in V$, then ℓ' must contain v.

Suppose for a contradiction that (ii) is false, so assume that there exists a second image of ℓ under the line map. Precisely, assume there exists a line ℓ^* so that if a point $\mathbf{v} \in V$ is to the left (right) of ℓ, then it is to the left (right) of both ℓ' and ℓ^*. However, this implies that the region of points to the right of ℓ and to the left of ℓ^* does not contain any images under f of points in V. This contradicts the fact that V is dense in $\mathbb{R}^2$, and f is a bijection.

To prove (iii), assume that f respects the line $\hat{\ell}$, and $\hat{\ell}'$ is the image of $\hat{\ell}$ under the line map, and suppose that ℓ and $\hat{\ell}$ are not parallel. Then ℓ and $\hat{\ell}$ divide the plane into four regions, consisting of points that are to the left or right of ℓ and to the left or right of $\hat{\ell}$. Now suppose, by contradiction, that ℓ' and $\hat{\ell}'$ are parallel. Then ℓ' and $\hat{\ell}'$ divide the plane into three regions. Thus, there is one combination, for example, to the left of ℓ' and to the right of $\hat{\ell}'$, which is an impossibility. Since V is dense in $\mathbb{R}^2$, all four regions formed by ℓ and $\hat{\ell}$ contain points of V, which gives a contradiction. To prove the converse, apply the analogous argument to f^{-1}.

For (iv), suppose that f is a step-isometry and ℓ is a line respected by f, with image ℓ'. Let $\hat{\ell}$ be an integer parallel of ℓ. Let $\mathbf{a} \cdot \mathbf{x} = r$ be the equation defining ℓ, and $\mathbf{a} \cdot \mathbf{x} = r + z$ the equation defining $\hat{\ell}$, where $z \in \mathbb{Z}$. Without loss of generality, assume $z > 0$. Let $\mathbf{a}' \cdot \mathbf{x} = r'$ be the equation of ℓ'. We claim that the line $\hat{\ell}$ must be respected, and its image under f is the line $\hat{\ell}'$ with equation $\mathbf{a}' \cdot \mathbf{x} = r' + z$.

Suppose, by contradiction, that there exists a point $\mathbf{w} \in V$ with image $\mathbf{w}' = f(\mathbf{w})$ so that $\mathbf{a} \cdot \mathbf{w} < r + z$, while $\mathbf{a}' \cdot \mathbf{w}' > r' + z$. Now choose $\mathbf{u} \in V$ so that $d_P(\mathbf{u}, \mathbf{w}) < z$ and $\mathbf{a} \cdot \mathbf{u} < r$. Since V is dense in $\mathbb{R}^2$ and the distance between ℓ and $\hat{\ell}$ equals z, we can choose such a $\mathbf{u}$. Let $\mathbf{u}' = f(\mathbf{u})$. Since f respects ℓ, we have that $\mathbf{a}' \cdot \mathbf{u}' < r'$. Since $\mathbf{w}$ and $\mathbf{u}$ lie on opposite sides of ℓ' and $\hat{\ell}'$, we conclude that $d_P(\mathbf{u}', \mathbf{w}') > d_P(\ell', \hat{\ell}') = z$. This contradicts the fact that f is a step-isometry. See Figure 3. □

Proof of Lemma 5. Let $\mathbf{v}_1, \mathbf{v}_2 \in V$, and let ℓ be the line through $\mathbf{v}_1$ and $\mathbf{v}_2$, and let ℓ' be the line through $\mathbf{v}_1' = f(\mathbf{v}_1)$ and $\mathbf{v}_2' = f(\mathbf{v}_2)$. Let $\mathbf{a} \cdot \mathbf{x} = r$ be the equation of ℓ and $\mathbf{a}' \cdot \mathbf{x} = r'$ the equation of ℓ'. We will first show that ℓ must be respected by f, and its image under the line map is ℓ'.

For a contradiction, assume that there exist points $\mathbf{u}, \mathbf{w} \in V$ so that $\mathbf{u}, \mathbf{w}$ are both on the right of ℓ, while $\mathbf{u}' = f(\mathbf{u})$ lies to the right of ℓ' and $\mathbf{w}' = f(\mathbf{w})$ to the left of ℓ'. More precisely, assume that $\mathbf{a} \cdot \mathbf{u} > r$ and $\mathbf{a} \cdot \mathbf{w} > r$, while $\mathbf{a}' \cdot \mathbf{u} > r'$ and $\mathbf{a}' \cdot \mathbf{w} < r'$.

Let $M > 0$ be an integer for which the following properties hold.

- (i) $M > d(\mathbf{u}, \mathbf{w})$.
- (ii) If the unit ball P is enlarged by a factor M and placed so that its centre is to the right of ℓ' and its boundary contains $\mathbf{w}'$, so that the tangent to this M-ball at $\mathbf{w}'$ is parallel to ℓ', then $\mathbf{v}_1'$ and $\mathbf{v}_2'$ must be inside the polygon. See Figure 4.

Note that (ii) can be achieved because d is a smooth metric, and thus, every slope is the slope of a tangent of P. See Figure 4.

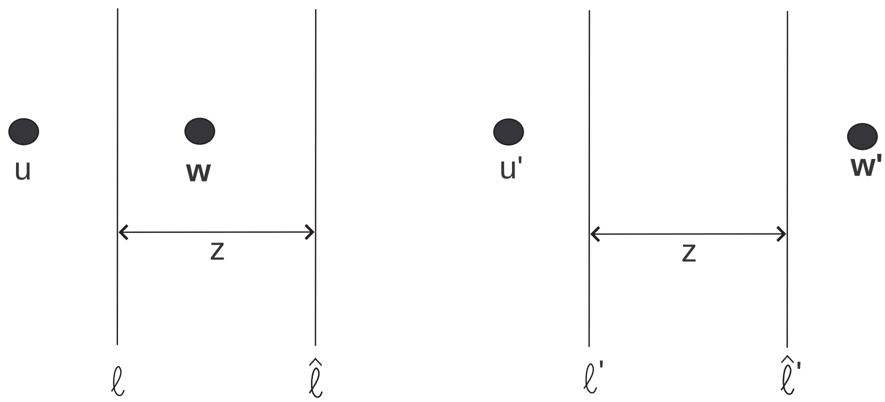

FIGURE 3. The distance between $\mathbf{u}'$ and $\mathbf{w}'$ must be greater than z, while the distance between $\mathbf{u}$ and $\mathbf{w}$ is less than z. This leads to a contradiction.

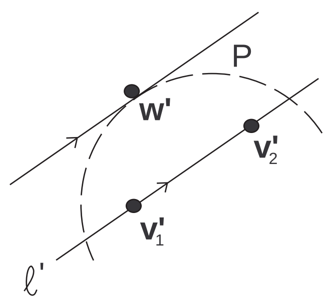

FIGURE 4. Illustration of property (ii) in the definition of M. The dotted line corresponds to the border of the M-ball.

Choose $\mathbf{p} \in V$ so that

$$\lfloor d(\mathbf{v}_1, \mathbf{p}) \rfloor = \lfloor d(\mathbf{v}_2, \mathbf{p}) \rfloor = M,$$

and

$$\lfloor d(\mathbf{u}, \mathbf{p}) \rfloor, \lfloor d(\mathbf{q}, \mathbf{w}) \rfloor < M.$$

Since $M > d(\mathbf{u}, \mathbf{w})$ and V is dense we may choose such a $\mathbf{p}$.

Since f is a step-isometry we have that a similar statement must hold for $\mathbf{p}' = f(\mathbf{p})$, and $\mathbf{v}_1', \mathbf{v}_2', \mathbf{u}', \mathbf{w}'$. Thus, the interior of the ball of radius M around $\mathbf{p}'$ must contain both $\mathbf{u}'$ and $\mathbf{w}'$, but not $\mathbf{v}_1'$ or $\mathbf{v}_2'$. By our choice of M this is impossible.

To show the second part of the lemma, let $\hat{\ell}$ be a line through a point $\mathbf{v}_3 \in V$ and parallel to ℓ. Let $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{v}_3$ be the equation of $\hat{\ell}$. We will show that $\hat{\ell}$ must be respected, and that its image is the line $\hat{\ell}'$ with equation $\mathbf{a}' \cdot \mathbf{x} = \mathbf{a}' \cdot \mathbf{v}_3'$. Note first that by Lemma 4, item (iv), all integer parallels of ℓ must also be respected. Thus, we may assume that the distance between ℓ and $\hat{\ell}$ is smaller than 1, and that ℓ is to the right of $\hat{\ell}$. (We can replace ℓ be one of its integer parallels if this is not the case.)

Suppose, by contradiction, that there exists a point $\mathbf{w} \in V$ with image $\mathbf{w}' = f(\mathbf{w})$ so that $\mathbf{w}$ is to the left of $\hat{\ell}$, while $\mathbf{w}'$ is to the right of $\hat{\ell}'$. Let $m \in \mathbb{Z}$ be so that $d(\mathbf{w}, \mathbf{v}_3) \leq m$, and so that, if the unit ball P is enlarged by a factor m and placed so it touches ℓ and its centre is to the left of ℓ (so it intersects $\hat{\ell}$), and so that it contains $\mathbf{v}_3$, then this m-ball cannot contain $\mathbf{w}'$. Now choose a point $\mathbf{u} \in V$ so that $\lfloor d(\mathbf{u}, \mathbf{v}_3) \rfloor = m-1$, $\lfloor d_P(\mathbf{u}, \mathbf{w}) \rfloor \leq m-1$, and $\mathbf{a} \cdot \mathbf{u} \leq r - m$. Note that the last condition implies that $\mathbf{u}$ is to the left of the integer parallel of ℓ which is distance m to the left of ℓ. Since f is a step isometry, and since the integer parallels of ℓ must be respected, similar conditions must hold for $\mathbf{u}' = f(\mathbf{u})$. This implies that the m-ball centered at $\mathbf{u}$ has its center to the left of ℓ, it touches or does not intersect ℓ, and it contains $\mathbf{v}_3'$ and $\mathbf{w}'$. By the condition on m, this is impossible. □

Note that we defer the proof of Lemma 6 to Section 3.

Proof of Lemma 7. If d_P is a polygonal metric, then let $\mathcal{G}$ be the (finite) set of generators of P; that is, the set of normals to the sides of P. If d_P is a smooth metric, then let $\mathcal{G}$ be the set of generators $\mathbf{a}$ of P such that there is a line through two points of V which has $\mathbf{a}$ as its normal vector. Since V is representative, $\mathcal{G}$ will be a generator set of P, and since V is countable, $\mathcal{G}$ is also countable. We will show, using induction, that for each set $B \subseteq V$, f must respect all lines in $\mathcal{L}(B, \mathcal{G})$.

Lemmas 5 and 6 show that any line through a point in B and with normal in $\mathcal{G}$ must be respected. Lemma 4 (*iv*), shows that any integer parallel of these lines must also be respected. This shows that f must respect all lines in $\mathcal{L}_0(B, \mathcal{G})$, and thus, establishes the base case of the induction.

Let $\sigma : \mathcal{G} \to \mathbb{R}^2$ be the function that defines the line map, so that for each line $\ell \in \mathcal{L}_0(B, \mathcal{G})$ through a point $\mathbf{b} \in B$, where ℓ has equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{b}$, the image of ℓ under the line map has as its equation $\mathbf{a} \cdot \mathbf{x} = \sigma(\mathbf{a}) \cdot f(\mathbf{b})$. By Lemma 4, item (*iii*), parallel lines that must be respected have images that are also parallel. Thus, any line which must be respected by f and which has normal vector $\mathbf{a}$, must have as its image a line with normal vector $\sigma(\mathbf{a})$. By item (*ii*) of the same lemma, if ℓ contains point $\mathbf{b}$, then its image contains $f(\mathbf{b})$. This shows that σ is well defined.

Consider the set $\mathcal{G}' = \{\sigma(\mathbf{a}) : \mathbf{a} \in \mathcal{G}\}$. By Lemma 6, if d_P is a polygonal metric, all lines parallel to one of the sides of P are mapped to lines that are again parallel to the sides of P, and thus, $\mathcal{G}' = \mathcal{G}$. If d_P is a smooth metric, then $\mathcal{G}$ is a countable dense subset of the set of all vectors from the origin to a point on the boundary of P. It is easy to see that $\mathcal{G}'$ must also be dense. Assume, to the contrary, that there exist two vectors $\mathbf{a}, \mathbf{a}'$ so that no vector in $\mathcal{G}'$ is "between" $\sigma(\mathbf{a})$ and $\sigma(\mathbf{a}')$. Fix $\mathbf{b} \in B$. Let ℓ and ℓ' be the lines through $\mathbf{b}$ with normal vectors $\mathbf{a}$ and $\mathbf{a}'$. Then, since $\mathcal{G}$ is dense, there must be a line ℓ^* through $\mathbf{b}$ which separates ℓ from ℓ'. This line must have a slope between the slopes of $\mathbf{a}$ and $\mathbf{a}'$. Thus, the image of ℓ^* must separate the images of ℓ and ℓ' and have a slope between that of $\sigma(\mathbf{a})$ and $\sigma(\mathbf{a}')$, which contradicts our assumption. Therefore, whether d_P is polygonal or smooth, in both cases $\mathcal{G}'$ is a generator set for P.

We will show by induction that f must respect all lines in $\mathcal{L}(B, \mathcal{G})$, and that σ defines the line map as stated. Precisely, we will recursively define a function $f^* : W \to \mathbb{R}$, where W is the set of all intersection points of lines in $\mathcal{L}(B, \mathcal{G})$, such that for any line $\ell \in \mathcal{L}(B, \mathcal{G})$ with equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{w}$ ($\mathbf{a} \in \mathcal{G}$, $\mathbf{w} \in W$), its image ℓ' is the line with equation $\sigma(\mathbf{a}) \cdot \mathbf{x} = \sigma(\mathbf{a}) \cdot f^*(\mathbf{w})$.

To simplify notation, we will use $\mathcal{L}_n$ to denote $\mathcal{L}_n(B,\mathcal{G})$. For all $k \geq 0$, let

$$\mathcal{L}_{\leq k} = \bigcup_{i=1}^{k} \mathcal{L}_i,$$

and let W_k be the set of all intersection points of lines in $\mathcal{L}_{\leq k}$. For the inductive step, assume the statement holds for a fixed $k \geq 0$. Precisely, assume that f must respect all lines in $\mathcal{L}_{\leq k}$ and that $f^* : W_k \to \mathbb{R}$ is defined so that the statement above holds. For any line $\ell \in \mathcal{L}_{\leq k+1}$ with equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{w}$ ($\mathbf{a} \in \mathcal{G}$, $\mathbf{w} \in W$), we define ℓ' to be the line with equation $\sigma(\mathbf{a}) \cdot \mathbf{x} = \sigma(\mathbf{a}) \cdot f^*(\mathbf{w})$. The induction hypothesis says that for any $\ell \in \mathcal{L}_{\leq k}$, ℓ' is its image under the line map. We will show that the same holds for $\ell \in \mathcal{L}_{k+1}(B)$.

We extend f^* to W_{k+1} by taking the image under f^* of the intersection of two lines in $\mathcal{L}_{\leq k}$ to be the intersection of the images of the lines under the line map. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathcal{G}$ be such that no two are linearly dependent. Suppose that $\ell_1, \ell_2 \in \mathcal{L}_{\leq k}$. Let ℓ_3 be a line in $\mathcal{L}_{k+1} \setminus \mathcal{L}_{\leq k}$ formed by the intersection of ℓ_1 and ℓ_2. Moreover, assume ℓ_i has normal vector $\mathbf{a}_i$, for $i = 1, 2, 3$. In this proof, we will say that a point $\mathbf{p}$ is to the right (left) of a line with equation $\mathbf{a} \cdot \mathbf{x} = t$ if $\mathbf{a} \cdot \mathbf{p} > t$ ($\mathbf{a} \cdot \mathbf{p} < t$). See Figure 5 for a visualization of the proof.

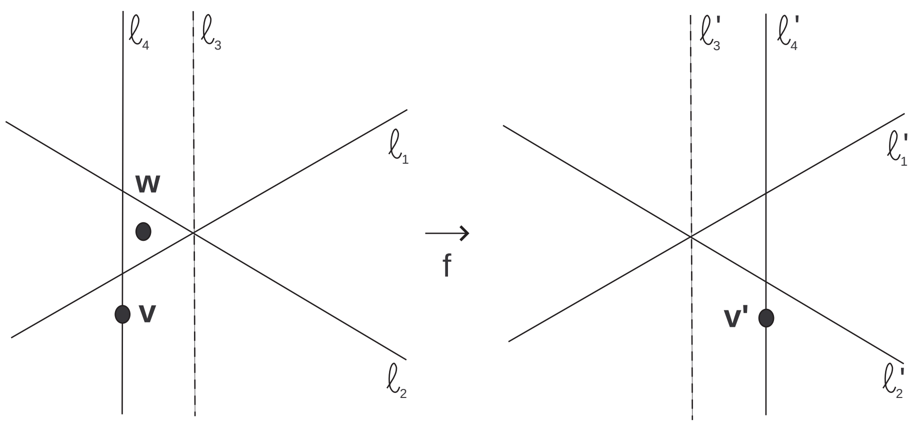

FIGURE 5. The point $f(\mathbf{w})$ does not exist.

Assume by contradiction that f does not respect ℓ_3. Precisely, assume there exists $\mathbf{v} \in V$ which is to the left of ℓ_3 such that $\mathbf{v}' = f(\mathbf{v})$ is to the right of the line ℓ_3'. Let ℓ_4 be the unique line parallel to ℓ_3 and through $\mathbf{v}$. By the base case, f must respect lines in $\mathcal{L}_0$, and so must respect ℓ_4. As V is dense in $\mathbb{R}^2$, we may choose $\mathbf{w} \in V$ so that $\mathbf{w}$ is to the right of ℓ_4, left of ℓ_1, and left of ℓ_2. See Figure 5. But then $f(\mathbf{w})$ must be to the right of ℓ_4', to the left of ℓ_1', and to the left of ℓ_2', which is a contradiction. $\square$

Proof of Lemma 8. Let $\mathcal{G}$ be a countable generator set for P which contains at least three vectors that are pairwise non-parallel. Let $B = \{\mathbf{p}, \mathbf{q}\}$, and fix $\mathbf{a} \in \mathcal{G}$, where $\mathbf{a} \cdot (\mathbf{p} - \mathbf{q}) = r \in (0, 1)$.

Without loss of generality, assume that $\mathbf{q} = \mathbf{0}$. Fix $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathcal{G}$ so that no two are linearly dependent. Consider the triangular lattice formed by the lines generated by the $\mathbf{a}_i$ and the

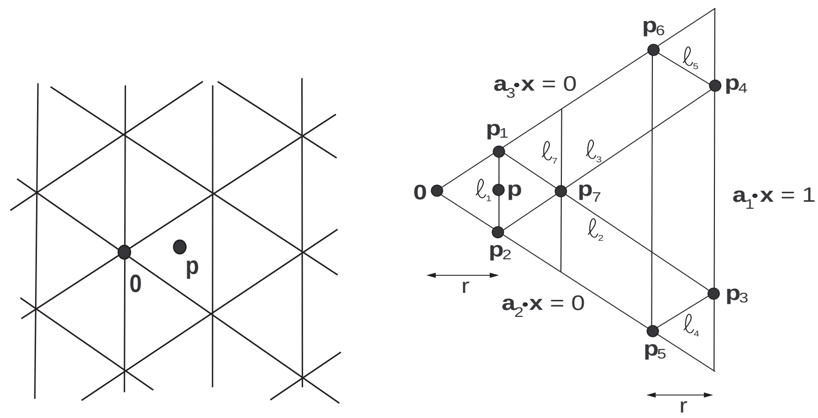

FIGURE 6. Left: A triangular lattice. Right: Generating the line $\mathbf{a}_3 \cdot \mathbf{p}_2 = 1 - r$.

point $\mathbf{0}$ in $\mathcal{L}_0(\{\mathbf{0}\}, \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\})$; that is, all lines with equations $\mathbf{a_i} \cdot \mathbf{x} = z$, where $i \in \{1, 2, 3\}$ and $z \in \mathbb{Z}$. See the left figure in Figure 6.

Consider the triangle that contains $\mathbf{p}$. We assume that this triangle is framed by the lines with equations $\mathbf{a}_1 \cdot \mathbf{x} = 1$, $\mathbf{a}_2 \cdot \mathbf{x} = 0$ and $\mathbf{a}_3 \cdot \mathbf{x} = 0$, as shown in Figure 6. (The proof can easily be adapted to cover all other possibilities.) By definition, the line ℓ_1 through $\mathbf{p}$ with normal $\mathbf{a}_1$ is part of $\mathcal{L}_0(B, \mathcal{G})$. The line ℓ_1 has as equation $\mathbf{a}_1 \cdot \mathbf{x} = r$, where $r = \mathbf{a}_1 \cdot \mathbf{p}$.

The line ℓ_1 intersects the two sides of the triangle in $\mathbf{p}_1$ and $\mathbf{p}_2$. Following the recursive definition, this implies that $\mathcal{L}_1(B, \mathcal{G})$ contains the line ℓ_2 through $\mathbf{p}_1$ with normal $\mathbf{a}_2$. Precisely, ℓ_2 has equation $\mathbf{a}_2 \cdot \mathbf{x} = \mathbf{a}_2 \cdot \mathbf{p}_1$. Similarly, $\mathcal{L}_1(B, \mathcal{G})$ contains the line ℓ_3 through $\mathbf{p}_2$ which has equation $\mathbf{a}_3 \cdot \mathbf{x} = \mathbf{a}_3 \cdot \mathbf{p}_2$. See the right figure in Figure 6.

The lines ℓ_2 and ℓ_3 intersect the third side of the triangle in $\mathbf{p}_3$ and $\mathbf{p}_4$, generating two lines ℓ_4 and ℓ_5 in $\mathcal{L}_2(B)$ with equations $\mathbf{a}_3 \cdot \mathbf{x} = \mathbf{a}_3 \cdot \mathbf{p}_3$ and $\mathbf{a}_2 \cdot \mathbf{x} = \mathbf{a}_2 \cdot \mathbf{p}_4$, respectively. The lines ℓ_4 and ℓ_5 intersect with the sides of the triangle in $\mathbf{p}_5$ and $\mathbf{p}_6$, generating one line ℓ_6 in $\mathcal{L}_3(B, \mathcal{G})$ with equation $\mathbf{a}_1 \cdot \mathbf{x} = \mathbf{a}_1 \cdot \mathbf{p}_5 = \mathbf{a}_1 \cdot \mathbf{p}_6$.

By the comparison of similar triangles, we obtain that

$$\mathbf{a}_1 \cdot \mathbf{p}_5 = \mathbf{a}_1 \cdot \mathbf{p}_6 = 1 - \mathbf{a}_1 \cdot \mathbf{p} = 1 - r.$$

Now the parallel lines $\mathbf{a}_1 \cdot \mathbf{x} = r + z_2, -r + z_2$, $z_2 \in \mathbb{Z}$, may be generated from all similar triangles in the lattice in an analogous fashion.

To complete the proof, consider that the lines ℓ_2 and ℓ_3 intersect in point $\mathbf{p}_7$, which generates a line ℓ_7 with normal $\mathbf{a}_1$ as indicated in the right figure in Figure 6. Since the triangle formed by $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_7$ is half of a parallelogram formed by $\mathbf{0}, \mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_7$, it follows that ℓ_7 has equation $\mathbf{a}_1 \cdot \mathbf{x} = 2r$. This process can be repeated to obtain all the lines $\mathbf{a}_1 \cdot \mathbf{x} = z_1 r + z_2$, $z_1, z_2 \in \mathbb{Z}$.

If r is irrational, then the set $\{z_1 r + z_2 : z_1, z_2 \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ (this is a result from folklore which can be proved by using the pigeonhole principle). That completes the proof of the lemma. $\square$

Proof of Lemma 9. Fix $\mathbf{q} \in V$ and $\mathbf{a} \in \mathcal{G}$. Define $S(\mathbf{a}) = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{a} \cdot (\mathbf{v} - \mathbf{q}) \in \mathbb{Q}\}$. The set $S(\mathbf{a})$ is the union of a countable number of lines, and thus, has measure zero in $\mathbb{R}^2$. Let

$$S = \bigcup_{\mathbf{a} \in \mathcal{G}} S(\mathbf{a}).$$

Then S is the countable union of sets of measure zero, so S has measure zero. Since V is representative, there must exist a point $\mathbf{p} \in V$ such that $V \notin S$, and thus, for all $\mathbf{a} \in \mathcal{G}$, $\mathbf{a} \cdot (\mathbf{p} - \mathbf{q})$ is irrational.

Let $B = \{\mathbf{p}, \mathbf{q}\}$. Let W be the set of all intersection points in $\mathcal{L}(B, \mathcal{G})$, and let σ and f^* be as given by Lemma 7. Fix $\mathbf{a} \in \mathcal{G}$, and let $r = \mathbf{a} \cdot (\mathbf{q} - \mathbf{p})$ and $r' = \sigma(\mathbf{a})\mathbf{a} \cdot (\mathbf{q}' - \mathbf{p}')$. By Lemma 8, for any $z_1, z_2 \in \mathbb{Z}$ the line ℓ with equation $\mathbf{a} \cdot \mathbf{x} = z_1 r + z_2$ is in $\mathcal{L}(B, \mathcal{G})$, and by Lemma 4, this line must be respected. Following the proof of Lemma 8, it can be easily deduced that the image of ℓ must have as its equation $\sigma(\mathbf{a}) \cdot \mathbf{x} = z_1 r' + z_2$. Namely, each of the points $\mathbf{p}_i$ used in the proof and in Figure 2 is defined by the intersection of previously given lines. Their image $f^*(\mathbf{p}_i)$ will be similarly defined by the intersection of the images of those lines. Thus, the images again follow a layout as in Figure 6 (right), but in this case the distance between the images of the reference points $\mathbf{0}$ and $\mathbf{p}$ equals r'.

We now claim that $r = r'$. Assume that this is not the case. Then there exist integers $z_1, z_2 \in \mathbb{Z}$ so that $z_1 r + z_2 < 1$ and $z_1 r' + z_2 > 1$. Choose a point $\mathbf{v} \in V$ so that $z_1 r + z_2 < \mathbf{a} \cdot (\mathbf{v} - \mathbf{q}) < 1$. Thus, $\mathbf{v}$ lies to the right of the line with equation $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = z_1 r + z_2$, and to the left of the line with equation $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 1$. Thus, its image $\mathbf{v}' = f(\mathbf{v})$ must lie to the right of the line with equation $\sigma(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{p}') = z_1 r' + z_2$, and to the left of the line with equation $\sigma(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{p}') = 1$. Hence, $z_1 r' + z_2 < \sigma(\mathbf{a}) \cdot \mathbf{v} < 1$, which is a contradiction. $\square$

4. Respecting lines when P is a polygon; proof of Lemma 6

In this section, we consider norm-derived metrics on $\mathbb{R}^2$ whose shape is a polygon. Specifically, fix $d \in \Omega$ so that d is a polygonal metric which is not a box metric, and let P be the shape of d. Let V be a representative set in $\mathbb{R}^2$ and let $f : V \to V$ be a step-isometry. Since P is a polygon, its generator set is finite. Let $\mathcal{G}$ be the generator set of P where each direction is represented by only one vector, so we have that $P = \{\mathbf{x} : \forall\, \mathbf{a} \in \mathcal{G},\ -1 \leq \mathbf{a} \cdot \mathbf{x} \leq 1\}$, and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}$,

$$d(\mathbf{x}, \mathbf{y}) = \max\{|\mathbf{a} \cdot (\mathbf{x} - \mathbf{y})| : \mathbf{a} \in \mathcal{G}\}.$$

Recall that we use $F(P, \mathbf{a})$ to denote the face of P with normal $\mathbf{a} \in \mathcal{G}$, where $\mathbf{a}$ is pointing away from the centre of P. We will show that any line through a point $\mathbf{v} \in V$ and parallel to one of the sides of P must be respected, and its image is a line through $\mathbf{v}' = f(\mathbf{v})$ parallel to one of the sides of P.

Before we prove Lemma 6, we need the following technical lemma. The lemma can be understood as follows. Given a point $\mathbf{p} \in \mathbb{R}$, and a line ℓ through $\mathbf{p}$ with normal vector $\mathbf{a} \in \mathcal{G}$, it is straightforward to define two polygons which are both similar to P, so that ℓ is the only line through $\mathbf{p}$ which separates the two polygons. Namely, consider two copies of P side-by-side; that is, they intersect in a face defined by $\mathbf{a}$, place them such that $\mathbf{p}$ lies on the shared face, and then move them apart so that they do not intersect, but no other line through $\mathbf{p}$ with normal vector in $\mathcal{G}$ fits between the two polygons. However, to define these separating polygons one must be able to precisely define the distances.

The following lemma shows that separating polygons can also be defined if only rounded distances are given. This is achieved by making the polygons very large. Before stating the lemma, we introduce the following notation for the rounded distance. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, define

$$D(\mathbf{x}, \mathbf{y}) = \lfloor d_P(\mathbf{x}, \mathbf{y}) \rfloor .$$

Lemma 11. *There exist positive integers M and m such that, given any points*

$$\mathbf{p}, \mathbf{z}, \mathbf{z}^*, \mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^* \in \mathbb{R}^2$$

which satisfy the conditions (3) below, there exists a unique line through $\mathbf{p}$ parallel to one of the sides of P which separates the sets $\{\mathbf{x} : d(\mathbf{x}, \mathbf{z}) \leq M\}$ and $\{\mathbf{x} : d(\mathbf{x}, \mathbf{z}^*) \leq M\}$.

$$\begin{aligned} D(\mathbf{x}, \mathbf{x}^*) &= D(\mathbf{y}, \mathbf{y}^*) = 0, \\ D(\mathbf{x}, \mathbf{p}) &= D(\mathbf{x}^*, \mathbf{p}) = D(\mathbf{y}, \mathbf{p}) = D(\mathbf{y}^*, \mathbf{p}) = m, \\ D(\mathbf{x}, \mathbf{y}) &= D(\mathbf{x}^*, \mathbf{y}^*) = 2m, \\ D(\mathbf{z}, \mathbf{x}) &= D(\mathbf{z}, \mathbf{y}) = D(\mathbf{z}^*, \mathbf{x}^*) = D(\mathbf{z}^*, \mathbf{y}^*) = M - 1, \\ D(\mathbf{z}, \mathbf{p}) &= D(\mathbf{z}^*, \mathbf{p}) = M, \\ D(\mathbf{z}, \mathbf{z}^*) &= 2M. \end{aligned} \tag{3}$$

Proof. Let m and M be positive integers that satisfy the following conditions. Note that the conditions impose only lower bounds, and thus, m and M can always be chosen so that the conditions hold by choosing them sufficiently large.

(*a*) Let $\theta_{\min}$ be the minimum angle between any two vectors in $\mathcal{G}$. We must choose m so that $m > 1/\sin(\theta_{\min})$.

(*b*) Let m be chosen such that, for all $\mathbf{a} \in \mathcal{G}$, the shortest path along the boundary of P from a point on the line ℓ^+ with equation $\mathbf{a} \cdot \mathbf{x} = 1/m$, and a point on the line ℓ^- with equation $\mathbf{a} \cdot \mathbf{x} = -1/m$ has length at most 1 (where length is measured according to metric d). See Figure 7.

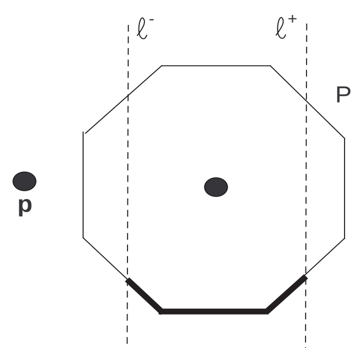

FIGURE 7. Condition (b): the length of the bold path along the boundary is at most $1/m$.

(*c*) The integer M is such that, for any $\mathbf{a} \in \mathcal{G}$, the region strictly between the lines ℓ^- with equation $\mathbf{a} \cdot \mathbf{x} = 1 - 1/M$, and the line ℓ^0 with equation $\mathbf{a} \cdot \mathbf{x} = 1$ (an extension of the face $F(P, \mathbf{a})$), contains no vertex of P. See Figure 8.

(*d*) The integers M and m are such that, for any $\mathbf{a} \in \mathcal{G}$, and lines ℓ^- and ℓ^0 as defined under (c), the following holds. Let $\mathbf{q}$ be a vertex of P on ℓ^0, and let $\mathbf{v}$ be the point on the intersection of the boundary of P and ℓ^- which is closest to $\mathbf{q}$. Then $d(\mathbf{v},\mathbf{q}) \leq \frac{m-1}{M}$. See Figure 8.

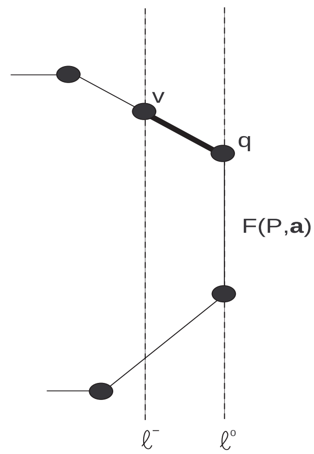

FIGURE 8. Condition (c): no vertices of P lie between ℓ^- and ℓ^0. Condition (d): limits the length of the bold part $\mathbf{v}$–$\mathbf{q}$ of the boundary of P.

(*e*) The integers M and m are so that $M > m > 2$, and M is large enough so that points can be chosen so that conditions (3) hold.

Let $\mathbf{p}, \mathbf{z}, \mathbf{z}^*, \mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^* \in \mathbb{R}^2$ be such that the conditions in (3) hold. Let

$$P_{\mathbf{z}} = B_M(\mathbf{z}) \text{ and } P_{\mathbf{z}^*} = B_M(\mathbf{z}^*).$$

See Figure 9. Hence, $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$ are the polygons, similar to P, which form the balls of radius M around $\mathbf{z}$ and $\mathbf{z}^*$, respectively. We will show first that there exists a line through $\mathbf{p}$ parallel to one of the sides of P which separates $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$.

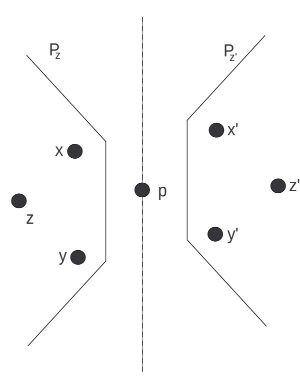

FIGURE 9. The balls $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$.

Let $\mathbf{a} \in \mathcal{G}$ be the vector that determines $d(\mathbf{z}, \mathbf{z}^*)$ so that

$$d(\mathbf{z}, \mathbf{z}^*) = \mathbf{a} \cdot (\mathbf{z}^* - \mathbf{z}),$$

and for all $\mathbf{a}_{\mathcal{G}} \in \mathcal{G}$,

$$|\mathbf{a}_{\mathcal{G}} \cdot (\mathbf{z}^* - \mathbf{z})| \leq d(\mathbf{z}, \mathbf{z}^*).$$

This implies that the line segment connecting $\mathbf{z}$ and $\mathbf{z}^*$ intersects $P_{\mathbf{z}}$ in $F(P_{\mathbf{z}}, \mathbf{a})$ and $P_{\mathbf{z}^*}$ in $F(P_{\mathbf{z}^*}, -\mathbf{a})$, as shown in Figure 10.

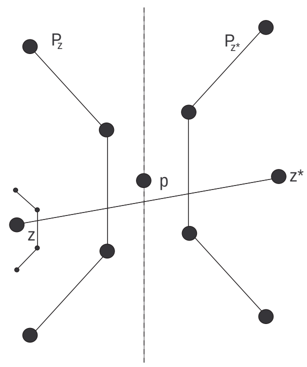

FIGURE 10. The line segment connecting $\mathbf{z}$ and $\mathbf{z}^*$ intersects $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$ in parallel faces.

Let ℓ_1 and ℓ_2 be the lines extending $F(P_{\mathbf{z}}, \mathbf{a})$ and $F(P_{\mathbf{z}^*}, -\mathbf{a})$, respectively. Precisely, ℓ_1 is the line with equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{z} + M$, and ℓ_2 has equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{z}^* - M$. Since $D(\mathbf{z}, \mathbf{z}^*) = \lfloor d(\mathbf{z}, \mathbf{z}^*) \rfloor = 2M$ we have that $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$ do not intersect, and thus, ℓ_1 lies to the left of ℓ_2. If $\mathbf{p}$ lies between ℓ_1 and ℓ_2, then the line with equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{p}$ separates $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$, as in Figure 9, and we are done.

Suppose now, by contradiction, that this is not the case. See Figure 11. Without loss of generality, suppose that $\mathbf{p}$ lies to the left ($\mathbf{z}$-side) of ℓ_1. By definition $\mathbf{x}^*$ and $\mathbf{y}^*$ lie inside $P_{\mathbf{z}^*}$, and thus, to the right of ℓ_2. Since $d(\mathbf{x}, \mathbf{x}^*) < 1$ and $d(\mathbf{y}, \mathbf{y}^*) < 1$, we must have that $\mathbf{x}$ and $\mathbf{y}$ both lie to the right ($\mathbf{z}^*$-side) of the line ℓ_3, which is an integer parallel of ℓ_2 defined by $\mathbf{a} \cdot \mathbf{x} = \mathbf{a} \cdot \mathbf{z}^* - M - 1$. Note that ℓ_3 is such that ℓ_2 and ℓ_3 are distance 1 apart in the polygon metric d. Since $d(\mathbf{p}, \mathbf{z}^*) < M + 1$, $\mathbf{p}$ must lie to the right of ℓ_3.

Consider $P_{\mathbf{p}} = B_m(\mathbf{p})$, the m-ball around $\mathbf{p}$. Let $\mathbf{v}$ be the point on the intersection of the boundary of $P_{\mathbf{z}}$ and ℓ_3 and let $\mathbf{q}$ be the vertex of $P_{\mathbf{z}}$ which is an endpoint of $F(P_z, \mathbf{a})$ (see Figure 6). (Following the definition, there are actually two choices for both $\mathbf{q}$ and $\mathbf{v}$; in each case, we choose the point closest to $\mathbf{p}$.) By condition (c) on M, no vertex of $P_{\mathbf{z}}$ lies between ℓ_3 and ℓ_1, and thus, the line segment from $\mathbf{q}$ to $\mathbf{v}$ is part of the boundary of $P_{\mathbf{z}}$. By condition (d), this line segment has length, measured according to d, at most $m - 1$.

Since $d(\mathbf{p}, \mathbf{z}) \leq M + 1$, the distance from $\mathbf{p}$ to the nearest point on the boundary of $P_{\mathbf{z}}$ is at most 1. Moreover, since $d(\mathbf{v}, \mathbf{q}) \leq m - 1$ this implies that there is a path of length at most m from $\mathbf{p}$ to $\mathbf{v}$ and from $\mathbf{p}$ to $\mathbf{q}$, respectively. Thus, $d(\mathbf{v}, \mathbf{p}) \leq m$ and $d(\mathbf{v}, \mathbf{q}) \leq m$, and so $\mathbf{v}$ and $\mathbf{q}$ lie inside $P_{\mathbf{p}}$, and, by convexity, so does the entire piece of the boundary of $P_{\mathbf{z}}$ between $\mathbf{q}$ and $\mathbf{z}$.

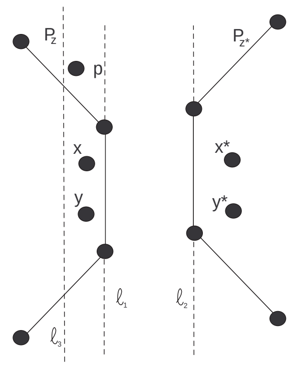

FIGURE 11. The point $\mathbf{p}$ must lie to the right of ℓ_3.

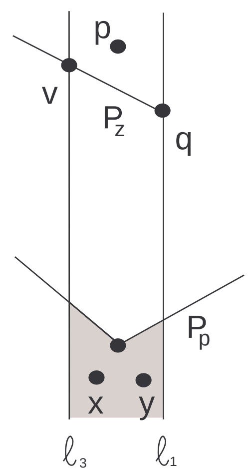

FIGURE 12. The position of $\mathbf{x}$ and $\mathbf{y}$. The shaded area is the only area between lines ℓ_1 and ℓ_3 outside $P_{\mathbf{p}}$ and inside $P_{\mathbf{z}}$.

We therefore have that there is only one connected region between ℓ_1 and ℓ_3 outside $P_{\mathbf{p}}$ and inside $P_{\mathbf{z}}$. See shaded area in Figure 12. Hence, $\mathbf{x}$ and $\mathbf{y}$ must both lie in the unique region outside $P_{\mathbf{p}}$ and between ℓ_1 and ℓ_3. In particular, they both lie on the same side of $P_{\mathbf{p}}$. However, since $d(\mathbf{x},\mathbf{p}) < m+1$ and $d(\mathbf{y},\mathbf{p}) < m+1$, $\mathbf{x}$ and $\mathbf{y}$ are both within distance 1 (in polygon distance) of $P_{\mathbf{p}}$. By condition (2), the length of the piece of the boundary of $P_{\mathbf{p}}$ between lines ℓ_1 and ℓ_3, measured in polygon distance, is at most m. Therefore,

$$d_P(\mathbf{x},\mathbf{y}) \leq m+2 < 2m,$$

which contradicts the assumption about the choice of $\mathbf{x}$ and $\mathbf{y}$. Hence, $\mathbf{p}$ lies between ℓ_1 and ℓ_2 as in Figure 9, and thus, the line ℓ with equation $\mathbf{a}\cdot\mathbf{x} = \mathbf{a}\cdot\mathbf{p}$ separates $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$.

Next, we prove that the line ℓ is unique with the given property. Suppose for a contradiction that there are two distinct lines ℓ^1 and ℓ^2, with normal vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathcal{G}_P$, respectively, so that both ℓ^1 and ℓ^2 contain $\mathbf{p}$ and separate $P_{\mathbf{z}}$ and $P_{\mathbf{z}^*}$. In particular, both lines must separate $\mathbf{x}$ and $\mathbf{x}^*$. Now, by the conditions in (3), $d(\mathbf{x},\mathbf{x}^*) < 1$, $d(\mathbf{p},\mathbf{x}) \geq m$ and $d(\mathbf{p},\mathbf{x}^*) \geq m$. Thus, the angle ϕ between the line segments $\mathbf{px}$ and $\mathbf{px}^*$ is such that $\sin(\phi) < 1/m$. The angle between ℓ^1 and ℓ^2 must be smaller than the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$, but by condition (a) the angle between any two vectors in $\mathcal{G}$ has sine value at least $1/m$. This gives a contradiction. □

Proof of Lemma 6. Assume without loss of generality that the origin $\mathbf{0} \in V$, and that $f(\mathbf{0}) = (\mathbf{0})$. Fix $\mathbf{a} \in \mathcal{G}$. Let M and m be as in Lemma 11, let $\mathbf{p} = \mathbf{0}$, and choose $\mathbf{z}, \mathbf{z}^*, \mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*$ such that $\|\mathbf{z}\|_P = \mathbf{a} \cdot \mathbf{z}$ and $\|\mathbf{z}^*\|_P = -\mathbf{a} \cdot \mathbf{z}^*$, and the conditions (3) of Lemma 11 hold. Hence, the sets $P_{\mathbf{z}} = B_M(\mathbf{z})$ and $P_{\mathbf{z}^*} = B_M(\mathbf{z}^*)$ are separated by the line ℓ with equation $\mathbf{x} \cdot \mathbf{a} = 0$.

Since f is a step-isometry, and conditions (3) only refer to rounded distances, the conditions (3) also hold for the images $f(\mathbf{p}) = \mathbf{0}, f(\mathbf{z}), f(\mathbf{z}^*), f(\mathbf{x}), f(\mathbf{x}^*), f(\mathbf{y})$, and $f(\mathbf{y}^*)$. Therefore, by Lemma 11 there exists a vector $\mathbf{a}' \in \mathcal{G}$ so that the line ℓ' with equation $\mathbf{a}' \cdot \mathbf{x} = 0$ separates the M-balls around $f(\mathbf{z})$ and $f(\mathbf{z}^*)$. Precisely, $\mathbf{a}' \cdot f(\mathbf{z}) < 0$ and $\mathbf{a}' \cdot f(\mathbf{z}^*) > 0$.

Claim: The map f is consistent with the line ℓ with equation $\mathbf{x} \cdot \mathbf{a} = 0$, and the image of ℓ under the line map is the line ℓ' with equation $\mathbf{a}' \cdot \mathbf{x} = 0$.

Proof of Claim. Fix any $\mathbf{w} \in V$ to the left of ℓ, so $\mathbf{a} \cdot \mathbf{w} < 0$. Choose integer $\overline{M} \geq M + 1$, and points $\bar{\mathbf{z}}$ and $\bar{\mathbf{z}}^*$ so that $D(\mathbf{w}, \bar{\mathbf{z}}) \leq \bar{M}$ and $\bar{\mathbf{z}}, \bar{\mathbf{z}}^*, \mathbf{x}, \mathbf{x}^*, \mathbf{y}, \mathbf{y}^*, \mathbf{p}$ satisfy conditions (3) with M replaced by $\overline{M}$. Moreover, the new points are chosen so that $P_{\mathbf{z}}$ is contained in $P_{\bar{\mathbf{z}}} = B_{\overline{M}}(\bar{\mathbf{z}}$ and $P_{\mathbf{z}^*}$ is contained in $P_{\bar{\mathbf{z}}^*} = B_{\overline{M}}(\bar{\mathbf{z}}^*)$.

By definition, the sets $P_{\bar{\mathbf{z}}})$ and $P_{\bar{\mathbf{z}}^*}$ are separated by the line ℓ, and $\mathbf{w} \in P_{\bar{\mathbf{z}}}$. By Lemma 11 there must be a line $\hat{\ell}$ through $f(\mathbf{p}) = \mathbf{0}$ with normal in $\mathcal{G}$ which separates the $\overline{M}$-balls around $f(\bar{\mathbf{z}})$ and $f(\bar{\mathbf{z}}^*)$. The line $\hat{\ell}$ also separates the M-balls around $f(\mathbf{z})$ and $f(\mathbf{z}^*)$, and thus, $\hat{\ell}$ line must be ℓ'. Therefore, $f(\mathbf{w})$ must lie on the left (that is, the $f(\mathbf{z})$-side) of ℓ'. □

By the claim, we can now define $\sigma : \mathcal{G} \to \mathcal{G}$ by taking $\sigma(\mathbf{a})$ to be the normal to the image under the line map of the line with equation $\mathbf{a} \cdot \mathbf{x} = 0$. By Lemma 4, σ is well-defined and a bijection.

By a similar argument, for every point $\mathbf{p} \in V$ we can define such a map σ. Moreover, by Lemma 4 (iii), parallel lines have parallel images under the line map. Hence, the map σ is independent of the choice of $\mathbf{p}$. □

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