

A NOTE ON ORIENTATIONS OF THE INFINITE RANDOM GRAPH

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ABSTRACT. We answer a question of P. Cameron's by giving examples of 2^{\aleph_0} many non-isomorphic acyclic orientations of the infinite random graph with a topological ordering that do not have the pigeonhole property. Our examples also embed each countable linear ordering.

A graph is *n-existentially closed* or *n-e.c.* if for each n -subset S of vertices, and each subset T of S (possibly empty), there is a vertex not in S , joined to each vertex of T and no vertex of $S \setminus T$. The infinite random graph, written R , is the unique (up to isomorphism) countable graph that is *n-e.c.* for all $n \geq 1$. For more on the infinite random graph, the reader is directed to [2, 3].

The infinite random graph is intimately related to a certain vertex partition property. A graph G has the *pigeonhole property*, written (\mathcal{P}) , if for every partition of the vertices of G into two nonempty parts, the subgraph induced by some one of the parts is isomorphic to G . This property was introduced by P. Cameron in [2], who in [3] classified the countable graphs with (\mathcal{P}) ; there are only four up to isomorphism: the graph with one vertex, the countably infinite clique and its complement, and R . In particular, R is the unique countable 1-e.c. graph that has (\mathcal{P}) . The pigeonhole property may be easily generalized to any relational structure. The countable tournaments with (\mathcal{P}) were classified in [1]; there are \aleph_1 many: the countable ordinal powers of ω and their reversals, and the countably infinite random tournament. A problem of [1] that has resisted solution is the following.

Problem 1: Classify the countable oriented graphs with (\mathcal{P}) .

As proved in [1], a countable oriented graph with (\mathcal{P}) that is neither a tournament nor the infinite random oriented graph O , must be an orientation of R . Cameron [4] was the first to notice that any such

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orientation must be acyclic (that is, contains no directed cycles), have infinitely many sources or infinitely many sinks, and admits a homomorphism into a countable ordinal.

A *topological order* of the vertices of an oriented acyclic graph $D = (V, E)$ is a linear order \preceq on V such that if $(x, y) \in E$, then $x \preceq y$. As an intermediate step towards the solution of Problem 1, Cameron [4] posed the following problem.

Problem 2: Are there 2^{\aleph_0} (that is, cardinality of the real numbers) many non-isomorphic acyclic orientations of the infinite random graph with a topological ordering that do not have (\mathcal{P}) ?

We say that an orientation of R as described in Problem 2 is *bad*.

The goal of this short note is to answer Problem 2 affirmatively. We actually prove a stronger assertion, as stated in the following theorem, which we think is of interest in its own right.

Theorem 1. *There are 2^{\aleph_0} many non-isomorphic bad orientations of R that embed each countable linear ordering.*

We consider only countable simple graphs and oriented graphs, which we refer to as orgraphs. Directed edges are written (x, y) and we say that x is *joined to* y and y is *joined from* x . If (x, y) is a directed edge in an orgraph, then we forbid (y, x) to be a directed edge. If G is a graph or orgraph, then $V(G)$ is the set of vertices of G ; the set $E(G)$ are the edges of G if G is a graph, and the directed edges of G if G is an orgraph. If $B \subseteq V(G)$, then we write $G \upharpoonright B$ for the subgraph or suborgraph induced by B ; if H is an induced subgraph or suborgraph of G , then we write $H \leq G$. We write $G \cong H$ if G and H are isomorphic. We say that G *embeds* in H if G is isomorphic to an induced subgraph or suborgraph of H . If D is an orgraph, then the *graph* of D , written $G(D)$, is the graph with vertices $V(D)$ and with edge set the symmetric closure of $E(D)$. A vertex u is a *source* in D if for every $v \in V(D)$, we have that $(v, u) \notin E(D)$; a *sink* is defined dually.

Proof of Theorem 1. For each integer $i \geq 3$, let L_i be the i -vertex linear order, and let $\Phi = \{L_i : i \geq 3\}$. Fix X , an infinite, co-infinite subset of Φ , and enumerate $X = \{L_{i_1}, L_{i_2}, \dots\}$ and $\Phi \setminus X = \{L_{j_1}, L_{j_2}, \dots\}$.

Define an orgraph $D(X)$ inductively, as follows. As we proceed with the induction, each vertex will be assigned exactly one colour, red or blue. Once a vertex has been assigned a colour, it will have that colour throughout the induction. If an induced suborgraph H has each of its vertices red (blue), then we say that H is red (blue). The orgraph $D(X)_1$ consists of two disjoint vertices with no directed edges between

them, with one vertex red and the other blue. The subgraph $Blue(1)$ is this blue vertex. Assume that $D(X)_n$ is defined, finite, and if $n \geq 2$, then $D(X)_{n-1} \leq D(X)_n$. We will also assume that there is at least one blue vertex in $D(X)_n$, and the subgraph $Blue(n)$ induced by the set of all the blue vertices of $D(X)_n$ is a finite linear order. The orgraph $D(X)_{n+1}$ is defined in four stages.

If M is a linear order, then a vertex is called *special for M* if x is joined to the first and last vertices of M only, and x is not joined from any vertex of M . For each red copy L of any one of $L_{i_1}, L_{i_2}, \dots, L_{i_n}$ in $D(X)_n$ whose initial vertex is *not a source* in $D(X)_n$, add a new red vertex x_L that is special for L . The resulting orgraph is called $D(X)'_{n+1}$.

For each pair $u, v \in Blue(n)$ so that $u < v$ in the linear order $Blue(n)$, add a new blue vertex b_{uv} so that $u < b_{uv} < v$. The vertices b_{uv} are not directed to or from any red vertex. Observe that there are $\binom{|Blue(n)|}{2}$ new blue vertices of the form b_{uv} added at this stage. After these blue vertices are added, add a new blue vertex a strictly less all the blue vertices, and a new blue vertex z strictly greater than all the blue vertices; the vertices a and z are not joined to or from any red vertex. The orgraph with these additional blue vertices $Blue(n+1) = \{a, z\} \cup \{b_{uv} : u, v \in Blue(n), u < v\}$ is called $D(X)'_{n+1, blue}$.

To $D(X)'_{n+1, blue}$ add a disjoint red copy of $L_{i_{n+1}}$ with an additional red vertex x_{n+1} that is special for $L_{i_{n+1}}$, and add a disjoint red copy of $L_{j_{n+1}}$ that has an additional red vertex y_{n+1} that is joined to the initial vertex of $L_{j_{n+1}}$ only and is not joined from any vertex of $L_{j_{n+1}}$. The resulting orgraph is called $D(X)''_{n+1}$.

Consider the finite undirected graph $G = G(D(X)''_{n+1})$. For each subset S of $V = V(G)$, and each subset T of S (possibly empty), add a new red vertex which is joined only to vertices of T . Orient the edges so that if a vertex $x \notin V$ is joined to $y \in V$, then (y, x) is a directed edge (in other words, edges are directed *out of G*). Give G back its orientation from $D(X)''_{n+1}$. This new orgraph is called $D(X)_{n+1}$. Observe that $Blue(n+1) \leq D(X)_{n+1}$ forms a linear order.

Define

$$D(X) = \bigcup_{n \geq 1} D(X)_n.$$

Hence, $D(X)$ is the union of the chain

$$D(X)_1 \leq D(X)_2 \leq D(X)_3 \leq \dots$$

It follows that $H = G(D(X)) \cong R$, since H is n -e.c. for all $n \geq 1$. To see this, fix $S \subseteq V(H)$ and T a subset of S . Then $S \subseteq V(D(X)_m)$ for

some positive integer m . A vertex not in S joined to vertices of T but not $S \setminus T$ may be found in $V(D(X)_{m+1})$ (by construction of $D(X)_{m+1}$).

Let $Blue(X)$ be the subgraph induced by the blue vertices of $D(X)$. The orgraph $Blue(X)$ is then the union of the chain $Blue(1) \leq Blue(2) \leq Blue(3) \leq \dots$ and is a linear order. It is not hard to see that $Blue(X)$ is dense and has no endpoints. Therefore, $Blue(X)$ is isomorphic to the order type of the rational numbers (see Theorem 2.8 of [5]). From the fact that the order type of the rational numbers embeds all countable linear orders (see Theorem 2.5 of [5]), it follows that $D(X)$ embeds each countable linear order.

We prove that $D(X)$ is acyclic by induction on n . Clearly, $D(X)_1$ is acyclic. Assume that $D(X)_n$ is acyclic. In $D(X)'_{n+1}$, the addition of no vertex x_L creates a directed cycle. No blue vertex added in $D(X)'_{n+1,blue}$ is joined to or from a red vertex; further, $Blue(n+1)$ is a linear order and hence, acyclic. Therefore, there is no directed cycle in $D(X)'_{n+1,blue}$. In $D(X)''_{n+1}$, the addition of neither x_{n+1} nor y_{n+1} creates a directed cycle. Since each z added in $V(D(X)_{n+1}) \setminus V(D(X)''_{n+1})$ is a sink in $D(X)_{n+1}$, there are no directed cycles in $D(X)_{n+1}$.

Define the subset A of the vertices of $D(X)$ to be all the “primed vertices”; in particular, A_1 is the unique red vertex of $D(X)_1$ that we name v , and $A_n = V(D(X)'_n) \setminus V(D(X)_{n-1})$ with $A = \bigcup_{n \geq 1} A_n$. The set A is *stable*, that is, there are no directed edges between vertices of A . To see this, we prove the following stronger claim.

Claim 1: The vertex v , and each vertex x_L added in at stage $D(X)'_n$, for each $n > 1$, are sources in $D(X)$.

To prove the claim, we prove by induction on s that v is a source in each $D(X)_s$ for $s \geq 1$, and each vertex x_L is a source in $D(X)_s$ for every $s \geq n$. Since the arguments used for the cases of v and x_L are similar, we consider only the argument for x_L .

If $s = n$, there is nothing to prove. Assume that x_L is a source in $D(X)_s$. If there is a vertex joined to x_L in $D(X)'_{s+1}$, then this vertex is of the form $x_{L'}$, where L' is a linear order in $D(X)_s$ whose initial vertex is not a source. But then x_L is a vertex of L' and so must be the initial vertex of L' , which gives a contradiction. Clearly, x_L is a source in $D(X)'_{n+1,blue}$, $D(X)''_{s+1}$, and $D(X)_{s+1}$.

For $n \geq 2$, define B_n to be the red vertices in $V(D(X)_n) \setminus V(D(X)'_n)$, and let $B = \bigcup_{n \geq 2} B_n$. Then A , $Blue(X)$, and B forms a partition of $V(D(X))$. We linearly order the vertices of $D(X)$ by first enumerating A with v as the first vertex, then listing the vertices of A_2 in some linear order, and then listing the vertices of A_3 in some order, and so on. Observe that each A_i is finite. Now include $Blue(X)$ in the linear

order, respecting the linear ordering of $Blue(X)$. Hence, each vertex of A is less every vertex of $Blue(X)$. Next, list the vertices of B by first listing B_1, B_2, \dots . Within each B_n , list first x_n , and then list vertices of L_{i_n} , respecting the linear order of L_{i_n} . Next list y_n and then the vertices of L_{j_n} , respecting the linear order of L_{j_n} . Finally, list the vertices of $V(D(X)_n) \setminus V(D(X)''_n)$ in any way. Hence, each vertex of $A \cup Blue(X)$ is less than every vertex of B . Name this linear ordering of $V(D(X))$ by $\mathcal{L} = (V(D(X)), \preceq)$.

The linear order \mathcal{L} is topological. To see this, note that this holds for vertices in A , since there are no directed edges between vertices of A . Since $Blue(X)$ is itself a linear order, the ordering there is topological. In B , the topological property is satisfied by our choice of ordering of the vertices of B . As the vertices of A are all sources in $D(X)$, and since A forms an initial segment of \mathcal{L} , we need only check the topological property for directed edges (c, d) , where either $c \in Blue(X)$ and $d \in B$ or $c \in B$ and $d \in Blue(X)$. However, since there are no directed edges (c, d) with c red and d blue, this follows immediately.

The linear order \mathcal{L} has order type $\omega + \eta + \omega$, where ω is the order type of the natural numbers, and η is the order type of the rational numbers. We now prove the following claim.

Claim 2: The orgraph $D(X)$ does not have (\mathcal{P}) .

Let $L \in X$ be fixed, and let L' be the copy of L added in disjointly to $D(X)''_m$ for some m . Recall that in $D(X)''_m$, the vertex x_m is special for L' . Let the initial vertex of L' be named a' . Let $C = \{x \in B \setminus \{x_m\} : x \preceq a'\}$. Observe that C is finite, since B has order type ω in \mathcal{L} . Furthermore, all the vertices of $D(X)$ different from x_m which are joined to a' are in $A \cup C$.

Let $A' = A \cup C \cup Blue(X)$, and let $B' = V(D(X)) \setminus A'$. The orgraph A' cannot be isomorphic to $D(X)$. Otherwise, $G = G(D(X) \upharpoonright (A'))$ would be isomorphic to R . Since A is a stable set and $Blue(X)$ is a linear order, G consists of the union of an infinite empty graph (the subgraph induced by the vertices of A), an infinite complete graph (the subgraph induced by the vertices of $Blue(X)$) and a finite graph (the subgraph induced by the vertices of C). But then G does not have property (\mathcal{P}) , which is a contradiction.

We show that $B'' = D(X) \upharpoonright B'$ cannot be isomorphic to $D(X)$ using the *back-and-forth* game or method, which in our case is a two player game of perfect information played in countably many steps on two countable orgraphs D_0 and D_1 . The players are named the *duplicator* and the *spoiler*. (The names come from the facts that the duplicator is trying to show the structures are alike, while the spoiler is trying to

show they are different.) A move consists of a choice of a vertex from either structure, and the spoiler makes the first move. The players take turns choosing vertices from the $V(D_i)$, so that if one player chooses a vertex from $V(D_i)$, the other must choose a vertex of $V(D_{i+1})$ (the indices are mod 2). Players cannot choose previously chosen vertices. After n rounds, this gives rise to a list of vertices $U_n = \{a_i : 1 \leq i \leq n\}$ from D_0 and $V_n = \{b_i : 1 \leq i \leq n\}$ from D_1 . The duplicator wins if for every $n \geq 1$, the subgraph induced by U_n is isomorphic to the subgraph induced by V_n . Otherwise, the spoiler wins. From this it follows that the duplicator has a winning strategy if and only if D_0 and D_1 are isomorphic. See [2] for more on the back-and-forth method.

Now the spoiler chooses in B'' the vertex x_m and the vertices of L' in succession. The duplicator must respond with $|V(L')|+1$ corresponding vertices in $D(X)$ that give rise to a linear order L'' and a vertex x''_m which is joined to the first vertex of L'' , which we name α . Since (x''_m, α) is a directed edge in $D(X)$, α is not a source and so there is a $z \in V(D(X)) \setminus \{x''_m\}$ that is joined to α . The spoiler can win in the next round by choosing z . To see this, note that the duplicator cannot now choose an appropriate vertex of B'' , since the spoiler already has chosen all the vertices of B'' which are joined to α . Claim 2 follows.

It is not hard to show that there are 2^{\aleph_0} many distinct infinite, co-infinite subsets of the natural numbers. To finish the proof of the theorem, we use this fact in conjunction with the following claim.

Claim 3: If $X \neq Y$, then $D(X) \not\cong D(Y)$.

Without loss of generality, there is an $L_i \in X \setminus Y$; name the first vertex of L_i a and the last vertex z . In $D(X)$, for *every* copy of L_i so that a is not a source, there is a vertex that is joined to *both* a and z . This is clear for the red vertices by construction, and for the blue vertices by the fact that $Blue(X)$ is a dense linear order without endpoints.

We show that this property fails for $D(Y)$, which will prove Claim 3. Consider a fixed copy of L_i added at some stage $D(Y)_m''$. Since (y_m, a) is a directed edge, the vertex a is not a source. However, there is no vertex $x' \in V(D(Y))$ which is joined to both a and z , although there is in $D(X)$. If there were such an x' in $D(Y)$, then x' must be a vertex added in at some stage $D(Y)'_{r+1}$. But the vertices of $D(Y)'_{r+1}$ not among the vertices of $D(Y)_r$ are special only for linear orders isomorphic to those in Y . Hence, if (x', a) is a directed edge, then (x', z) is *not* a directed edge, otherwise, $L_i \in Y$. (We are tacitly using here the fact that there is, up to isomorphism, exactly one linear order on n vertices, if n is a positive integer.) \square

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