

Burning a graph is hard^{*}

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Abstract. We introduce a new graph parameter called the burning number, inspired by contact processes on graphs such as graph bootstrap percolation, and graph searching paradigms such as Firefighter. The burning number measures the speed of the spread of contagion in a graph; the lower the burning number, the faster the contagion spreads. We provide algorithmic results on the burning number, and prove the corresponding graph decision problem is **NP**-complete even when restricted to graph classes such as planar graphs, disconnected graphs, or bipartite graphs.

1 Introduction

Suppose you were attempting to spread gossip, a meme, or some other social contagion on a social networking site such as Facebook or Twitter. Our assumptions, similar to those in the recent study on the spread of emotional contagion in Facebook [13], are that in-person interaction and nonverbal cues are not necessary for the spread of the contagion. Hence, agents in the network spread the contagion to their friends or followers, and the contagion propagates over time. If the goal was to minimize the time it took for the contagion to reach the entire network, then which users would you target with the contagion, and in which order? Related questions emerge in study of the spread of social influence, which is an active topic in social network analysis; see, for example, [7,11,12,15,16,3]. As a simplified, deterministic approach to these questions, we consider a new approach involving a graph process which we call *burning*. Burning is inspired by graph theoretic processes like Firefighting [4,5,6,9], graph cleaning [1], and graph bootstrap percolation [2].

Graph burning may be viewed as a one-player game, where the player attempts to burn all the nodes as quickly as possible. Throughout, we work with simple, undirected, and finite graphs. There are discrete time-steps or rounds. Each node is either *burned* or *unburned*; if a node is burned, then it remains in that state until the end of the game. Every round, the player chooses a node to burn. Once a node is burned in round t , each of its unburned neighbours becomes burned in round $t+1$. In every round, the player chooses one additional unburned node to burn (if such a node is available). The game ends when all nodes are

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burned. The *burning number* of a graph G , written by $b(G)$, is the minimum number of rounds needed for the game to end. For example, it is straightforward to see that $b(K_n) = 2$. However, even for paths computing the burning number is less trivial; in fact, $b(P_n) = \lceil n^{1/2} \rceil$ as we prove below in Theorem 2.

Burning may be viewed as a simplified model for the spread of social contagion in a social network such as Facebook or Twitter. The lower the value of $b(G)$, the easier it is to spread such contagion in the graph G . Suppose that in the process of burning a graph G , we eventually burned the whole graph G in k steps, and for each i , $1 \leq i \leq k$, we denote the node that we burn in the i -th step by x_i . We call such a node simply a *source of fire*. The sequence $(x_1, x_2, \dots, x_k)$ is called a *burning sequence* for G . With this notation, the burning number of G is the length of a shortest burning sequence for G ; such a burning sequence is referred to as *optimal*. For example, for the path P_4 with nodes v_1, v_2, v_3, v_4 , the sequence (v_2, v_4) is an optimal burning sequence. See Figure 1.

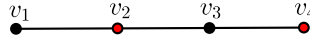


Fig. 1: Burning the path P_4 .

Note that for a G with at least two nodes, $b(G) \geq 2$. Unlike many graph parameters, the burning number of a disconnected graph G with components $G_1, G_2, \dots, G_t$, $t \geq 2$, does not necessarily follow the equality $b(G) = b(G_1) + b(G_2) + \dots + b(G_t)$. For example, let G be a disjoint union of t paths of order 2. One way to burn G is to burn exactly one node of the first $t - 1$ paths in the first $t - 1$ steps, and then burn the last P_2 in 2 steps. Hence, we have that $b(G) \leq (t - 1) + 2 = t + 1$, while the sum of the burning numbers of the components is $2t$.

In this extended abstract, we focus on determining the algorithmic complexity of the burning number of graphs. Specifically, we will consider the following algorithmic problem.

Problem: Burning Number

Instance: A finite simple graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$, and an integer $k \geq 2$.

Question: Is $b(G) \leq k$? In other words, does G contain a burning sequence $(x_1, x_2, \dots, x_k)$?

Our main result is that the Burning Number problem is **NP**-complete when $k \geq 3$ (it is in **P** if $k = 2$; see Theorem 5). Hence, burning a graph in an optimal fashion is “hard”.

Theorem 1. *Given a graph G and a positive integer $k \geq 3$, the decision problem $b(G) \leq k$ is **NP**-complete even when restricted to planar or disconnected graphs.*

As a corollary, we derive that the Burning Number problem is hard even on bipartite graphs.

Corollary 1. *Given a graph G and a positive integer $k \geq 3$, the decision problem $b(G) \leq k$ is **NP**-complete for bipartite graphs.*

The strategy of the proof of Theorem 1 uses a reduction from the planar 3-SAT problem. The proofs of these results are deferred to the next section. We apply some of the ideas that they were used in the reductions for proving the **NP**-completeness of the domination problem [10]. We finish the paper with discussion and open problems.

2 Proof of Theorem 1 and Corollary 1

We need some basic definitions from graph theory; for more background on graph theory, see [17]. If v is a node of a graph G , then the *eccentricity* of v is defined as $\max\{d(v, u) : u \in V(G)\}$. The *center* of G consists of the nodes in G with minimum eccentricity. Every node in the center of G is called a *central* node of G . The *radius* of G is the minimum eccentricity over the set of all nodes in G . The *diameter* of G is the maximum eccentricity over the set of all nodes in G . Given a positive integer k , the *k -th closed neighborhood* of v is defined to be the set $\{u \in V(G) : d(u, v) \leq k\}$ and is denoted by $N_k[v]$; we denote $N_1[v]$ simply by $N[v]$.

For $s \geq 3$, let $K_{1,s}$ denotes a *star*; that is, a complete bipartite graph with parts of order 1 and s . We call a graph obtained by a sequence of subdivisions starting from $K_{1,s}$ a *spider graph* SP . A subgraph of SP which is obtained by finitely many times subdividing an edge of $K_{1,s}$ is called an *arm* of SP (we think of the central node as belonging to each arm). If all arms of a spider graph are of the same length r , then we denote such a spider graph by $SP(s, r)$. If we join the central nodes of two spider graphs $SP(s, r)$ by adding an edge between them, then we call the obtained graph a *double spider graph*, and we denote it by $DSP(s, r)$. In a spider graph $SP(s, r)$ there is only one central node that is in fact corresponding to the central node of the initial star $K_{1,s}$. Similarly, in a double spider graph $DSP(s, r)$ there are exactly two adjacent central nodes that are corresponding to the central node of the initial copies of star graph $K_{1,s}$.

By a recursive argument, we may find the burning number of paths in polynomial time.

Theorem 2. *For every path P_n on n nodes, we have that $b(P_n) = \lceil n^{1/2} \rceil$.*

Proof. We first make the following observation. Suppose that $(x_1, x_2, \dots, x_k)$, where $k \geq 3$, is a burning sequence for a given graph G . For $1 \leq i \leq k$, the fire spread from x_i will burn only all the nodes within distance $k - i$ from x_i by the end of the k -th step. On the other hand, every node $v \in V(G)$ must be either a source of fire, or burned from at least one of the sources of fire by the end of the k -th step. In other words, any node of G that is not a source of fire

must be an element of $N_{k-i}[x_i]$, for some $1 \leq i \leq k$. Therefore, we can see that $(x_1, x_2, \dots, x_k)$ forms a burning sequence for G if and only if the following set equation holds:

$$N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \dots \cup N_0[x_k] = V(G). \quad (1)$$

From (1), and the fact that for a node v in a path, $|N_i[v]| \leq 2i + 1$ we derive that

$$\begin{aligned} & (2(k-1) + 2(k-2) + \dots + 2(1)) + k \\ &= 2 \left(\frac{k(k-1)}{2} \right) + k \\ &= k^2 \geq n. \end{aligned}$$

Since k is the minimum number satisfying this inequality, we conclude that $b(P_n) \geq \lceil n^{1/2} \rceil$.

Now let P_n have nodes $v_1, v_2, \dots, v_n$, and for $i = 1, \dots, k-1$, we choose $x_i = v_{n-i^2-i+2}$. For $i = k$, if $n \geq (k-1)^2 + k$, we take $x_k = v_{n-k^2-k+2}$; otherwise, we let $x_k = v_1$. Thus, we can burn P_n in exactly k steps by using the burning sequence $(x_1, x_2, \dots, x_k)$. Hence, $b(P_n) \leq k$.

Note that in the proof of Theorem 2 the burning sequence $(x_1, x_2, \dots, x_k)$ for paths may be found in polynomial time. The following theorem (proof omitted) gives us useful bounds for the burning number of a graph G .

Theorem 3. *For a connected graph G of radius r and diameter d we have that*

$$\lceil (d+1)^{1/2} \rceil \leq b(G) \leq r+1.$$

Note that the bounds in Theorem 3 are tight, since by Theorem 2, the lower bound is achieved by paths, and by Lemma 1 (1), the upper bound is achieved by spider graphs.

A subgraph H of a graph G is called an *isometric subgraph* if, for every pair of nodes in H denoted by u, v , $d_H(u, v) = d_G(u, v)$. For example, since there is a unique path between any two nodes of a tree, every subtree of a tree is an isometric subgraph. As another example, if G is a connected graph and P is a shortest path connecting two nodes of G , then P is an isometric subgraph of G . The following theorem, whose proof is omitted, is important for our proofs.

Theorem 4. *For an isometric subgraph H of a graph G , we have that $b(H) \leq b(G)$.*

We note that the inequality in Theorem 4 does not hold for non-isometric subgraphs. For example, let H be a path of order $n \geq 5$, and form G by adding a universal node to H . Then by Theorem 2, $b(H) = 3$, but $b(G) = 2$.

We can characterize the graphs with burning number 2 by the theorem below (with proof omitted).

Theorem 5. *A graph G satisfies $b(G) = 2$ if and only if G has order at least 2, and has maximum degree $n - 1$ or $n - 2$.*

From Theorem 5 we can determine if the burning number of a given graph G is equal to 2 or not in polynomial time. We now need the following lemma to prove Theorem 1.

Lemma 1. *1. For a spider graph $SP(s, r)$, with $s \geq r$, we have that $b(SP(s, r)) = r + 1$. Moreover, for $s \geq r + 2$, every optimal burning sequence of $SP(s, r)$ must start by burning the central node.*
2. For a double spider graph $DSP(s, r)$ with $s \geq r + 1$, we have that $b(DSP(s, r)) = r + 2$. Moreover, for $s \geq r + 2$, if u and v are the central nodes of $DSP(s, r)$, and in an optimal burning sequence $(x_1, x_2, \dots, x_{r+2})$ node x_1 is not a central node, then either x_1 is adjacent to u and $x_2 = v$, or x_1 is adjacent to v and $x_2 = u$.

As the proof of Lemma 1 (2) is analogous to the proof of item (1), we omit it owing to space considerations.

Proof. First, by Theorem 3, we know that $b(SP(s, r)) \leq r + 1$, as $SP(s, r)$ has radius r . Since $SP(r, r)$ is an isometric subgraph of $SP(s, r)$, with $s \geq r$, it suffices to show that $b(SP(r, r)) = r + 1$.

We need to prove that every burning sequence of $SP(r, r)$ is of length at least $r + 1$. Suppose otherwise; that is, there exist a burning sequence $(x_1, x_2, \dots, x_r)$ for $SP(r, r)$. First, we claim that in such a burning sequence there must be a source of fire selected from each arm of $SP(r, r)$. Otherwise, assume that there exist an arm of $SP(r, r)$ such that we do not burn directly any node of it as a source of fire throughout the process. Since the fire should burn every node in $SP(r, r)$, at some step, the fire should spread through the nodes of that arm, and this can happen, in the best case, in the second step and only from the central node (this follows since the only connection of each arm to the rest of the graph is through the central node). Consequently, after burning the center, the fire needs exactly $r + 1$ steps to burn every node on that arm, which is a contradiction. Hence, in this burning process we have to choose at least one source of fire from each arm. Now, we have two possibilities: either the center node is in the burning sequence or not.

If we choose the center node as one of the sources of fire, then by the Pigeonhole Principle, there must be an arm which we do not select any node of it (rather than the center node) as a source of fire throughout the process. But the fire must be spread to all nodes of such an arm. Thus, we need at least $r + 1$ steps, which is a contradiction. Thus, the existence of such a burning sequence is impossible.

If the center node is not taken as a source of fire, then, by above claim, we have to select exactly one node from each arm as a source of fire. In this way, when we are choosing the last source of fire from the last arm, every node on that arm should be burned before the last step, except for at most two nodes. Thus, we need at least $1 + (r - 1) + 1 = r + 1$ steps, which is again a contradiction.

Therefore, such a burning sequence does not exist, and every burning sequence for $SP(r, r)$ must be of length at least $r + 1$. Hence, $b(SP(r, r)) = r + 1$.

Now, suppose that $s \geq r + 2$, and there exist an optimal burning sequence of $SP(s, r)$ which does not start by burning the central node. Therefore, by the Pigeonhole Principle, one of the arms does not include any source of fire, except for possibly the central node, which can be a source of fire in other steps. Hence, we need at least $1 + (r + 1)$ steps for burning the nodes on that arm, which is a contradiction. Thus, every optimal burning sequence of $SP(s, r)$ starts by burning the central node, when $s \geq r + 2$.

We now state the following decision problem to prove the **NP**-completeness of the burning number problem. A graph G is *planar* if there is a drawing of G in the plane such that no two edges of G cross each other except at the end-points. Assume that $U = \{u_1, u_2, \dots, u_k\}$ is a finite set of Boolean variables. A 3-element set $C = \{x_1, x_2, x_3\}$ in which each x_i is an element of U like u_i or its negation (denoted by $\bar{u}_i$), and we do not have both u_i and $\bar{u}_i$ in the same time in C , is called a *clause* over U . Given set U and a collection of 3-element clauses over U such as $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$, we may construct a graph $G(\mathcal{C})$ as follows. For $1 \leq i \leq k$, let $E_i = \{u_i, \bar{u}_i\}$ be an edge between two nodes corresponding to variable $u_i \in U$ and its negation. We take $\mathcal{C} \cup \left(\bigcup_{i=1}^k E_i\right)$ as the set of nodes of $G(\mathcal{C})$. We join each u_i to any C_j satisfying $u_i \in C_j$. Similarly, we join each $\bar{u}_i$ to any C_j that $\bar{u}_i \in C_j$.

Problem: Planar 3-SAT

Instance: A finite set $U = \{u_1, u_2, \dots, u_k\}$ of Boolean variables and a collection $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ of 3-element clauses over U , such that the graph $G(\mathcal{C})$ as described above is planar.

Question: Is there any satisfying truth assignment for $\mathcal{C}$; that is, a truth assignment for the variables in U such that every clause in $\mathcal{C}$ contains at least one True assigned variable?

It is known that planar 3-SAT is **NP**-complete; see [14].

Proof (Proof of Theorem 1). The burning problem is in **NP** since it takes polynomial time to find the $(k-i)$ -th neighborhood of x_i , for $1 \leq i \leq k$, and therefore, to verify (1).

Now, let $U = \{u_1, u_2, \dots, u_k\}$ for some positive integer k , and let

$$\mathcal{C} = \{C_1, C_2, \dots, C_m\}$$

be a collection of 3-element clauses over U . We construct a graph $G'(\mathcal{C})$ based on $\mathcal{C}$ as follows. First, we consider the graph $G(\mathcal{C})$ as mentioned above with nodes $\mathcal{C} \cup \left(\bigcup_{i=1}^k E_i\right)$, where $E_i = \{u_i, \bar{u}_i\}$. Then we replace each edge $\{u_i, C_j\}$ and $\{\bar{u}_i, C_j\}$ in $G(\mathcal{C})$ with a path of length $k + 1 - i$ denoted by P_{ij1} and P_{ij2} , respectively. Second, we add a single new node s' . We join arbitrarily s' to one of the nodes in $\mathcal{C}$, and we attach to each node u_i and $\bar{u}_i$ a spider graph $SP(k + 2 - i, k - i)$ denoted by S_{i1} and S_{i2} (with centers u_i and $\bar{u}_i$), respectively.

Finally, we add a copy of a spider graph $SP(k+4, k+1)$ with center s to the obtained graph, by connecting one of the leaves of $SP(k+4, k+1)$ to a node of $\mathcal{C}$. The resulting graph is $G'(\mathcal{C})$; see Figure 2. It is evident that $|V(G'(\mathcal{C}))| = O(k^3 + mk^2 + m)$. The graph $G'(\mathcal{C})$ is planar as we have attached some copies of the spider graphs to a subdivision of $G(\mathcal{C})$ which is a planar graph. Note that $\mathcal{C}$ forms an independent set for $G'(\mathcal{C})$. Moreover, for $1 \leq i \leq k$, every edge E_i forms the center of a double spider graph $DSP(k-i+2, k-i)$, denoted by DS_i .

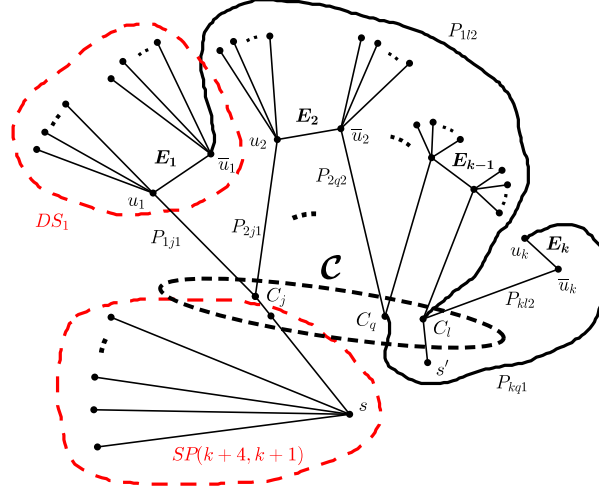


Fig. 2: A depiction of the graph $G'(\mathcal{C})$.

We want to show that $\mathcal{C}$ is satisfiable over U (of size k) if and only if $G'(\mathcal{C})$ has a burning sequence of size $k+2$. First, suppose that there is a satisfying truth assignment for $\mathcal{C}$. Take $S = (s, x_1, x_2, \dots, x_k, s')$ in which $x_i = u_i$ if u_i is True; otherwise, $x_i = \bar{u}_i$. We claim that S forms a burning sequence of size $k+2$ for $G'(\mathcal{C})$. To show this, we need to prove that by burning S in the specified order, every other node in $G'(\mathcal{C})$ will be burned after $k+2$ steps. First, we can easily see that for $1 \leq i \leq k$, all nodes in S_{i1} and S_{i2} are within distance $k-i+1$ from x_i , and they will be burned after $(k-i+1) + i + 1 = k+2$ steps. Since the x_i 's satisfy $\mathcal{C}$, each $C_j \in \mathcal{C}$ contains at least a True assigned variable like x_i , $1 \leq i \leq k$. Therefore, at the end of the $(k+2)$ -th step, C_j will be burned, by the fire spread from x_i through the path P_{ijl} , where $l \in \{1, 2\}$. All the internal nodes on each P_{ijl} are of distance at most $k-i+1$ from x_i , and hence, they will be burned after $(k-i+1) + i + 1 = k+2$ steps. Finally, by Lemma 1 (1), all the nodes on $SP(k+4, k+1)$ will be burned at step $k+2$ (after burning s in the first step).

Conversely, suppose that $G'(\mathcal{C})$ has a burning sequence of size $k+2$ denoted by $(y_1, y_2, \dots, y_{k+2})$. Since we have a copy of $SP(k+3, k+1)$ with the center

s , and by construction, here $SP(k+3, k+1)$ is an isometric subgraph of $G'(\mathcal{C})$, then by Theorem 3, we have $b(G'(\mathcal{C})) \geq k+2$. But by assumption, $G'(\mathcal{C})$ has already a burning sequence of length $k+2$. Hence, the sequence must be an optimal burning sequence. Since the only connection between $SP(k+3, k+1)$ and the rest of $G'(\mathcal{C})$ is through a path of length $k+2$ which is adjacent just to s , then by Lemma 1 (1) we conclude that y_1 must be equal to s .

Now the nodes in E_1 form the set of central nodes of a $DSP(k+1, k-1)$ as an isometric subgraph of $G'(\mathcal{C})$. Therefore, by Lemma 1 (2), we need at least $k+1$ steps to burn this $DSP(k+1, k-1)$. But we already chose $y_1 = s$, and $d(s, x) \geq k+3$ for each $x \in E_1$. On the other hand, the two nodes in E_2 are the central nodes of a $DSP(k, k-2)$ as an isometric subgraph of $G'(\mathcal{C})$. Thus, by Lemma 1 (2), we need at least k steps for burning this $DSP(k, k-2)$. Therefore, we have to choose y_3 from DS_2 , and we know that $d(x, y) \geq (k-1) + (k-2) \geq k$, for any $x \in E_1$ and $y \in E_2$. This implies that such a x and y cannot have common neighbours. Thus, by Lemma 1 (2), $y_2 \neq y_3$ and we conclude that y_2 must be the central node of a S_{1q} with $q \in \{1, 2\}$; that is, $y_2 = x$, for some $x \in E_1$.

Similarly, by induction, for each $2 \leq i \leq k-1$, the nodes in E_i are the central nodes of a $DSP(k-i+2, k-i)$ as an isometric subgraph of $G'(\mathcal{C})$. Thus, by Lemma 1 (2), we need at least $k-i+2$ steps for burning this $DSP(k-i+2, k-i)$. Therefore, for $1 \leq i \leq k-1$, we have to choose y_{i+1} from one of the S_{il} 's, where $q \in \{1, 2\}$. But we already have selected $y_1 = s, y_2, \dots, y_i$, and $d(s, x) \geq k+3$, for any $x \in E_i$. Further, $d(x, y) > k-i$, for any $x \in E_i$ and $y \in E_j$ with $1 \leq j \leq i-1$. On the other hand, the nodes in E_{i+1} are the central nodes of a $DSP(k-i+1, k-i-1)$ as an isometric subgraph of $G'(\mathcal{C})$. Thus, by Lemma 1 (2), we need at least $k-i+1$ steps for burning this $DSP(k-i+1, k-i-1)$. Therefore, we have to choose y_{i+2} from DS_{i+1} , and we know that $d(x, y) > k-i$, for any $x \in E_i$ and $y \in E_{i+1}$. It implies that such x 's and y 's cannot have common neighbours. Thus, by Lemma 1 (2) we have that $y_{i+1} \neq y_{i+2}$, and we conclude that y_{i+1} must be the central node of a S_{il} , where $l \in \{1, 2\}$; that is, $y_{i+1} = x$, for some $x \in E_i$.

After choosing $y_1 = s, y_2, \dots, y_k$, we consider y_{k+1} as follows. We know that E_k is an edge with two nodes and for burning an edge we need two steps. On the other hand, all nodes in the neighborhood of E_k are unburned so far, and there is no node among them adjacent to both nodes of E_k (which follows by the definition of 3-SAT where we assume u_i and its negation do not appear in any given clause), and we are left only with two steps. Therefore, we must have $y_{k+1} \in E_k$. Finally, since s' is of distance at least two from any burned node in the $(k+1)$ -th step, then we have that $y_{k+2} = s'$.

We know that, by our above discussion, all nodes in E_i 's, S_{il} 's, P_{ijl} 's (except for possibly the end points of some of P_{ijl} 's which are in $\mathcal{C}$), and $SP(k+3, k+1)$ will be burned after $k+2$ steps. Now, we have to consider $\mathcal{C}$. By assumption, $b(G'(\mathcal{C})) = k+2$, and we already proved that for $1 \leq i \leq k$, y_{i+1} must be selected from E_i . On the other hand, since E_i is an edge, and considering the construction of $G'(\mathcal{C})$, if $y_{i+1} = x \in E_i$, and P_{ijl} 's are the paths that connect x to the nodes in $\mathcal{C}$, then we can easily see that by the end of the $(k+2)$ -th step,

all nodes on P_{ijl} 's including their end points in $\mathcal{C}$ are burned. But for any other P_{ijr} , $r \neq l$, the end point of P_{ijl} which is a node in $\mathcal{C}$ is not burned. Therefore, for having $\mathcal{C}$ burned by the end of the $(k+2)$ -th step, there must be a selection of $y_2, y_3, \dots, y_{k+1}$ such that each $C_j \in \mathcal{C}$ will be burned by receiving the fire spread from y_i 's at the $(k+2)$ -th step. Hence, $\{y_i\}_{i=2}^{k+1}$ must be assigned True variables which satisfy $\mathcal{C}$.

The preceding discussion provides a reduction from planar 3-SAT problem to the burning problem on a connected planar graph. We may modify $G'(\mathcal{C})$ such that the resulting graph is disconnected, just by taking s' as an isolated node, and proof will be analogous. Hence, burning problem is **NP**-complete as well for disconnected graphs.

We are now ready to give our proof of Corollary 1.

Proof (Proof of Corollary 1). From Theorem 1, we know that for a graph G , and $k \geq 3$ the decision problem $b(G) \leq k$ is **NP**-complete. Now, we now show that there exists a reduction from this problem to the decision problem $b(B) \leq k+1$, where B is a bipartite graph.

Let G be a graph with node set $V(G) = \{v_1, v_2, \dots, v_n\}$. We construct a bipartite graph G' based on the structure of G as follows. First, we take two copies of the node set of G denoted by $X = \{v'_1, v'_2, \dots, v'_n\}$ and $Y = \{v''_1, v''_2, \dots, v''_n\}$. For $1 \leq i, j \leq n$, we join v'_i to v''_j if and only if, either $i = j$ or v_j is a neighbor of v_i in G . When we refer to a subset of nodes in X that is corresponding to the closed r -neighborhood of a node such as v in G , we denote it by $N_r^X[v]$. Similarly, we denote such a subset of Y by $N_r^Y[v]$. Then we add a copy of a spider graph $SP(n+k+1, k)$ with center s and we identify n leaves of it by nodes in Y . The resulted graph is called G' , and is bipartite. Also, $|V(G')| = n + k(n+k+1) + 1$, and we can easily see that G' is constructed in polynomial time from G . Moreover, since $SP(n+k+1, k)$ is an isometric subgraph of G' , and by Lemma 1 (1), we conclude that

$$b(G') \geq b(SP(n+k+1, k)) = k+1. \quad (2)$$

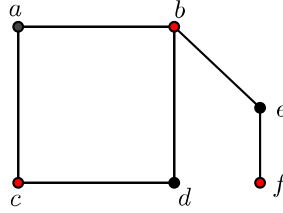


Fig. 3: The graph G .

For example, if G is the graph in Figure 3, then G' will be the graph in Figure 4. We need the following elementary lemma to complete the reduction.

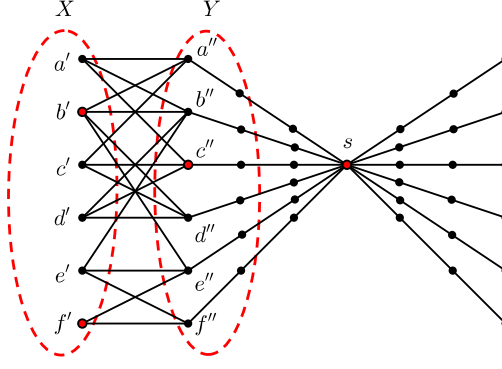


Fig. 4: The graph G' .

Lemma 2. *For any node $v' \in X$ in the above bipartite graph G' which is corresponding to node $v \in V(G)$ we have that $N_r[v'] = N_r^X[v]$, if r is even; otherwise, $N_r[v'] = N_r^Y[v]$. Similarly, for $v'' \in Y$ corresponding to $v \in V(G)$ if r is even, then $N_r[v''] = N_r^Y[v]$; otherwise, we have that $N_r[v''] = N_r^X[v]$.*

First, assume that $b(G) = k$, and $x_1, x_2, \dots, x_k$ is a burning sequence for G . We claim that $b(G') = k + 1$. By (2), to prove $b(G') = k + 1$ it suffices to show that $S = (y_0 = s, y_1 = x'_1, y_2 = x''_2, \dots, y_k = x''_k)$ if k is odd, and $S = (y_0 = s, y_1 = x''_1, y_2 = x'_2, \dots, y_k = x'_k)$ when k is even, is a burning sequence for G' . Clearly, by burning s in the first step, every node in $SP(n + k + 1, k)$, including the nodes in Y , will be burned after $k + 1$ steps. Thus, we have to show that by the end of the $(k + 1)$ -th step all nodes of X will be burned. If k is odd, then for each i that is odd, we selected y_i from X , and for each even i we selected y_i from Y . Thus, by Lemma 2, we can easily see that for $1 \leq i \leq k$ considering the parity of $k - i$, $N_{k-i}[y_i] = N_{k-i}^X[x_i]$ (based on the notation that we defined at the beginning of this proof). Similarly, using Lemma 2, we can conclude that if k is even, then for $1 \leq i \leq k$, $N_{k-i}[y_i] = N_{k-i}^X[x_i]$. Since, by assumption, $(x_1, x_2, \dots, x_k)$ is a burning sequence for G , then the $N_{k-i}[x_i]$'s form a covering for $V(G) = X$. Thus, X will be burned by the end of the $(k + 1)$ -th step. Therefore, $b(G') = k + 1$.

Now assume that $b(G') = k + 1$. We want to show that $b(G) \leq k$. By Lemma 1 (2), for burning the leaves of $SP(n + k + 1, k)$ that have no connection to the rest of G' except through s (and we have $k + 1$ number of such leaves), we have to start from s . Also, by burning s in the first step, every other node in $SP(n + k + 1, k)$ will be burned after $k + 1$ steps, so we can assume that the other sources of fire are selected from $X \cup Y$. Thus, every burning sequence for G' is of the form $(y_0 = s, y_1, y_2, \dots, y_k)$, in which $y_i \in X \cup Y$, for $1 \leq i \leq k$. Also, since all the nodes in X must be burned by the end of the $(k + 1)$ -th step, we conclude that the sets $N_{k-i}[y_i] \cap X$ form a covering for X . We know that for $1 \leq i \leq k$, either $y_i = v'_{j_i}$ or $y_i = v''_{j_i}$ for some $1 \leq j_i \leq n$. Therefore, by

Lemma 2, $N_{k-i}[y_i] \cap X = N_r^X[v_{j_i}]$, in which $r \leq k - i$ (in fact, either $r = k - i$ or $r = k - i - 1$, depending on whether $y_i \in X$ or $y_i \in Y$). Hence, $\{N_r^X[v_{j_i}]\}_{i=1}^k$ forms a covering for X . Thus, the sequence $(x_1 = v_{j_1}, x_2 = v_{j_2}, \dots, x_k = v_{j_k})$ forms a burning sequence for G . This implies that $b(G) \leq k$, and the proof follows.

3 Conclusions and open problems

Our main contribution is the algorithmic study of the new graph parameter, the burning number of a graph, written $b(G)$. The burning number measures how rapidly contagion spreads in a given graph. We proved that the decision problem for the burning number is **NP**-complete, even when restricted to graph classes such as planar graphs, disconnected graphs, or bipartite graphs. A tantalizing open problem is to determine the algorithmic complexity of the burning number of trees. We remark that by equation (1), burning number is related to domination since any burning sequence gives us a covering for the set of nodes of the graph in terms of the neighbourhoods. It is known that domination is in fact polynomial time for trees [10]. In contrast, Firefighting is **NP**-complete even for trees of maximum degree three [8].

A simple variation which leads to complex dynamics is to change the rules for nodes to burn. As in graph bootstrap percolation [2], the rules could be varied so nodes burn only if they are adjacent to at least r burned neighbors, where $r > 1$. We plan on studying this variation in future work.

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